

Finite-temperature form factors in the free Majorana theory

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Abstract

We study the large distance expansion of correlation functions in the free massive Majorana theory at finite temperature, alias the Ising field theory at zero magnetic field on a cylinder. We develop a method that mimics the spectral decomposition, or form factor expansion, of zero-temperature correlation functions, introducing the concept of “finite-temperature form factors”. Our techniques are different from those of previous attempts in this subject. We show that an appropriate analytical continuation of finite-temperature form factors gives form factors in the quantization scheme on the circle. We show that finite-temperature form factor expansions are able to reproduce expansions in form factors on the circle. We calculate finite-temperature form factors of non-interacting fields (fields that are local with respect to the fundamental fermion field). We observe that they are given by a mixing of their zero-temperature form factors and of those of other fields of lower scaling dimension. We then calculate finite-temperature form factors of order and disorder fields. For this purpose, we derive the Riemann-Hilbert problem that completely specifies the set of finite-temperature form factors of general twist fields (order and disorder fields and their descendants). This Riemann-Hilbert problem is different from the zero-temperature one, and so are its solutions. Our results agree with the known form factors on the circle of order and disorder fields.

1 Introduction

Quantum field theory (QFT) at finite temperature is a subject of great interest which has been studied from many viewpoints (see, for instance, [1]). The main goal of QFT is the reconstruction of a set of correlation functions in which the physical information provided by the theory is embedded. For instance, two-point correlation functions are related to response functions, which can be measured and which provide precise information about the dynamics of a physical system at thermodynamic equilibrium. For the purpose of comparison with experiment, it is important to know the influence of a non-zero temperature on correlation functions. In particular, both static (equal-time) and dynamical two-point correlation functions at finite temperature still need more accurate study.

In recent years, thanks to advances in experimental techniques allowing the identification and study of quasi-one-dimensional systems (see for instance [2, 3]), there has been an increased interest in calculating correlation functions in 1+1-dimensional integrable models of QFT (for applications of integrable models to condensed matter systems, see for instance the recent review [4]). Integrable models are of particular interest, because in many cases, matrix elements of local fields in eigenstates of the Hamiltonian, or form factors, can be evaluated exactly by solving an appropriate Riemann-Hilbert problem in the rapidity space [5, 6, 7, 8, 9]. At zero temperature, correlation functions are vacuum expectation values in the Hilbert space of quantization on the line. A useful representation of zero-temperature correlation functions is then provided by their form factor expansion (or spectral decomposition):

$$\begin{aligned} \langle \text{vac} | \mathcal{O}(x, \tau) \mathcal{O}(0, 0) | \text{vac} \rangle = & \sum_{k=0}^{\infty} \sum_{\epsilon_1, \dots, \epsilon_k} \int \frac{d\theta_1 \cdots d\theta_k}{k!} e^{-r \sum_j M(\epsilon_j) \cosh(\theta_j)} \times \\ & \times \langle \text{vac} | \mathcal{O}(0, 0) | \theta_1, \dots, \theta_k \rangle_{\epsilon_1, \dots, \epsilon_k}^{in} \langle \theta_1, \dots, \theta_k | \mathcal{O}(0, 0) | \text{vac} \rangle_{\epsilon_1, \dots, \epsilon_k}^{in} \end{aligned}$$

where τ is the Euclidean time and $r = \sqrt{x^2 + \tau^2}$. This expansion is obtained by inserting the resolution of the identity in terms of a basis of common eigenstates of the momentum operator and of the Hamiltonian, parameterized by the rapidity variables θ_j 's and by the particle types ϵ_j 's. Here $M(\epsilon)$ is the mass of a particle of type ϵ (and we took the basis of *in*-states). Using many of such insertions, form factor expansions can be obtained for multi-point correlation functions as well. This is a useful representation because it is a large-distance expansion, which is hardly accessible by perturbation theory, and which is often the region of interest in condensed matter applications. Also, form factor expansions in integrable models at zero temperature have proven to provide a good numerical accuracy for evaluating correlation functions in a wide range of energies, and combined to conformal perturbation theory give correlation functions at all energy scales (an early work on this is [10]).

In this paper we study the large-distance expansion of correlation functions in the free massive Majorana field theory on a line at finite temperature. This model of 1+1-dimensional quantum field theory occurs, for instance, as the off-critical scaling limit of the finite-temperature Ising quantum chain (with an infinite number of sites) in a transverse magnetic field, near to its quantum critical point. It also occurs as the off-critical scaling limit of the statistical Ising model on a cylinder, near to its thermal critical point; the circumference of the cylinder is the inverse temperature of the Majorana theory. For an exposition on these scaling limits and

for references of early works, see, for instance, the book [11]. In particular, in both of these examples, the fields of most interest are the twist fields associated to the \mathbb{Z}_2 symmetry of the Majorana theory [12, 13]. These fields are the scaling limit of the spin operator in the Ising chain (the Pauli matrix that enjoys a nearest-neighbor interaction) and of the spin variables in the statistical Ising model. There are two of these twist fields: the order field and the disorder field. They can be seen as representing the scaling limit of the same operator, but in different regimes (ordered and disordered) away from the critical point. When the mass is set to zero (at criticality), the theory reduces to the well-known Ising conformal field theory.

The correlation functions of twist fields exhibit non-trivial, “non-free” behaviors, and can be obtained as appropriate solutions to non-linear differential equations [14]. At finite temperature, or on the cylinder, these equations are partial differential equations in the coordinates on the cylinder [15, 16, 17], and do not offer a very useful tool for numerically evaluating correlation functions, neither for analyzing their large-distance behavior (short-distance behaviors, on the other hand, can always be analyzed by conformal perturbation theory [10]). Hence it is worth studying in more detail the large-distance expansion of correlation functions of twist fields. This study constitutes a first step towards generalizations to interacting integrable models. In particular, we will present a method that parallels the zero-temperature form factor expansion.

1.1 Correlation functions at finite temperature

At finite temperature, correlation functions are obtained by taking a statistical average of quantum averages, with Boltzmann weights e^{-LE} where E is the energy of the quantum state:

$$\langle\langle\mathcal{O}(x, \tau) \cdots\rangle\rangle_L = \frac{\text{Tr} [e^{-LH} \mathcal{O}(x, \tau)]}{\text{Tr} [e^{-LH}]} . \quad (1.1)$$

Here, L is the inverse temperature and H is the Hamiltonian. A consequence of the imaginary-time formalism [18] is the Kubo-Martin-Schwinger (KMS) identity [19, 20],

$$\langle\langle\mathcal{O}(x, \tau) \cdots\rangle\rangle_L = (-1)^f \langle\langle\mathcal{O}(x, \tau + L) \cdots\rangle\rangle_L \quad (1.2)$$

where f is the statistics of \mathcal{O} (it is 1 for fermionic operators and 0 for bosonic operators), and where the dots (\cdots) represent local fields at time τ and at positions different from x . Then, finite-temperature correlation functions can be seen as vacuum expectation values in the same model of QFT, but this time quantized on a circle of circumference L :

$$\langle\langle\mathcal{O}(x, \tau) \cdots\rangle\rangle_L = e^{i\pi s/2} {}_L\langle\text{vac}|\mathcal{O}_L(-\tau, x) \cdots|\text{vac}\rangle_L \quad (1.3)$$

where s is the spin of \mathcal{O} . The operator $\mathcal{O}_L(-\tau, x)$ is the corresponding operator acting on the Hilbert space on the circle, with space variable $-\tau$ (parameterizing the circle of circumference L) and Euclidean time variable x (going along the cylinder). The vector $|\text{vac}\rangle_L$ is the vacuum in the Hilbert space on the circle. There are two sectors in the quantization on the circle: Neveu-Schwartz (NS) and Ramond (R), where the fermion fields are anti-periodic and periodic, respectively, around the circle. The trace (1.1) with insertion of operators that are local with respect to the fermion fields naturally corresponds to the NS sector due to the KMS identity.

With insertion of twist fields, however, the sector may be changed to R or to a mixed sector by an appropriate choice of the associated branch cuts. The phase factor in (1.3) is in agreement with the fact that the operator \mathcal{O}_L is Hermitian on the Hilbert space on the circle, if the corresponding operator \mathcal{O} is Hermitian on the Hilbert space on the line.

1.2 Previous works

A straightforward application of the zero-temperature ideas would suggest a form factor expansion of the vacuum expectation values on the circle (1.3):

$${}_L\langle \text{vac} | \mathcal{O}_L(x, \tau) \mathcal{O}_L(0, 0) | \text{vac} \rangle_L = \sum_{k=0}^{\infty} \sum_{n_1, \dots, n_k} \frac{e^{\sum_j n_j \frac{2\pi i x}{L} - E_{n_1, \dots, n_k} \tau}}{k!} {}_L\langle \text{vac} | \mathcal{O}_L(0, 0) | n_1, \dots, n_k \rangle_L {}_L\langle n_1, \dots, n_k | \mathcal{O}_L(0, 0) | \text{vac} \rangle_L \quad (1.4)$$

where the eigenstates of the momentum operator and of the Hamiltonian on the circle are parametrized by discrete variables n_j 's. This form is valid for any integrable models on the circle; it can also obviously be generalized to other multi-point correlation functions. In general integrable models, the energy levels E_{n_1, \dots, n_k} are not known in closed form (although exact methods exist to obtain non-linear integral equations that define them: from thermodynamic Bethe ansatz techniques, from calculations à la Destri-de Vega and from the so-called BLZ program). Also the matrix elements of local fields ${}_L\langle \text{vac} | \mathcal{O}_L(0, 0) | n_1, \dots, n_k \rangle$ (form factors on the circle) still seem to be far from reach (however, see [21, 22, 23, 24]). Hence, this method does not seem to be applicable in general yet. Nevertheless, in the Majorana fermion theory with mass m , the energy levels are simply given by

$$E_{n_1, \dots, n_k} = \sum_{j=1}^k \sqrt{m^2 + \left(\frac{2\pi n_j}{L} \right)^2}$$

where $n_j \in \mathbb{Z} + \frac{1}{2}$ in the Neveu-Schwartz (NS) sector and $n_j \in \mathbb{Z}$ in the Ramond (R) sector (the sectors where the fermion field have anti-periodic and periodic conditions around the cylinder, respectively). Matrix elements of the primary order and disorder fields were calculated in [25], [26] in the lattice Ising model, and in a simpler way in [27] directly in the Majorana field theory using the free-fermion equations of motion and the “doubling trick”. The form factor expansion on the circle can then be used to obtain the leading exponentially decreasing behavior of static correlation functions in the quantum Ising chain, and, with slightly more work, to obtain the large-time dynamical correlation functions in the semi-classical regime [28]. This reproduces early results of [29] and [30].

A second set of ideas was suggested by Leclair, Lesage, Sachdev and Saleur in [31], starting from the representation (1.1). By partly performing the trace and by using general properties of matrix elements of local fields, the following representation for two-point functions in the

Majorana theory of mass m was obtained:

$$\begin{aligned} \langle\langle \mathcal{O}_1(x, \tau) \mathcal{O}_2^\dagger(0, 0) \rangle\rangle_L = \\ \sum_{k=0}^{\infty} \sum_{\epsilon_1, \dots, \epsilon_k = \pm} \int \frac{d\theta_1 \dots d\theta_k e^{\sum_{j=1}^k \epsilon_j (imx \sinh \theta_j - m\tau \cosh \theta_j)}}{k! \prod_{j=1}^k (1 + e^{-\epsilon_j mL \cosh \theta_j})} F_{\epsilon_1, \dots, \epsilon_k}^{\mathcal{O}_1}(\theta_1, \dots, \theta_k) (F_{\epsilon_1, \dots, \epsilon_k}^{\mathcal{O}_2}(\theta_1, \dots, \theta_k))^* \end{aligned} \quad (1.5)$$

where $F_{\epsilon_1, \dots, \epsilon_k}^{\mathcal{O}}(\theta_1, \dots, \theta_k)$ are matrix elements of \mathcal{O} in the Hilbert space on the line of the form $\langle \text{vac} | \mathcal{O} | \theta_{j_1}, \dots, \theta_{j_n}, \theta_{i_m} + i\pi, \dots, \theta_{i_1} + i\pi \rangle$, where all θ_j 's correspond to $\epsilon_j = (+)$, and all θ_i 's correspond to $\epsilon_i = (-)$.

This expression is better adapted to study dynamical correlation functions in real time $\tau = it$ than the form factor expansion on the circle. Another advantage of this idea is that, following methods similar to those used in [32] at zero temperature, one can obtain a Fredholm determinant representation for finite-temperature two-point functions, which can be useful for further analysis. Such a representation was obtained in [31] and was used in order to obtain a set of non-linear partial differential equations for the two-point function of twist fields. This lead to the known equations plus, in particular, an additional equation involving temperature derivatives. However, this additional equation does not follow from any of the other methods used in other works [15, 16, 17], and it was suggested not to hold in [17].

This idea also seems to allow more easily generalizations to interacting integrable models. Such a generalization was conjectured by Leclair and Mussardo in [33]. Saleur, however, pointed out that this generalization might not be correct [34], and the numerical calculation of Castro-Alvaredo and Fring in the scaling Lee-Yang model tended to agree with this incorrectness [35].

Note that verifications of the results of [31] were made for some fields in the complex (Dirac) free fermion and in the Federbush model in [35], and that agreement was found with known results in various limits. Leclair and Mussardo also proposed in [33], along similar lines, a formula for one-point functions of local operators in interacting integrable models, which, after some controversy, seem to be confirmed [36, 37, 38], although it is fair to say that more checks would be appropriate (on the subject of one-point functions from form factors, see also [39]).

In the works mentioned above, very little was said concerning finite-temperature correlation functions with more than two operators.

1.3 Our work

In the following, we will develop a scheme for obtaining large-distance representations of finite-temperature correlation functions in the Majorana theory of the form:

$$\begin{aligned} \langle\langle \mathcal{O}_1(x, \tau) \mathcal{O}_2^\dagger(0, 0) \rangle\rangle_L = \\ \sum_{k=0}^{\infty} \sum_{\epsilon_1, \dots, \epsilon_k = \pm} \int \frac{d\theta_1 \dots d\theta_k e^{\sum_{j=1}^k \epsilon_j (imx \sinh \theta_j - m\tau \cosh \theta_j)}}{k! \prod_{j=1}^k (1 + e^{-\epsilon_j mL \cosh \theta_j})} f_{\epsilon_1, \dots, \epsilon_k}^{\mathcal{O}_1}(\theta_1, \dots, \theta_k; L) (f_{\epsilon_1, \dots, \epsilon_k}^{\mathcal{O}_2}(\theta_1, \dots, \theta_k; L))^* \end{aligned} \quad (1.6)$$

with appropriate generically L -dependent functions $f_{\epsilon_1, \dots, \epsilon_k}^{\mathcal{O}}(\theta_1, \dots, \theta_k; L)$. We will see that this formula is valid:

- with both $\mathcal{O}_1, \mathcal{O}_2$ fields that are local with respect to the fermion fields, in the NS sector;
- with one of \mathcal{O}_1 or \mathcal{O}_2 a twist field, in a mixed R-NS or NS-R sector;
- with both $\mathcal{O}_1, \mathcal{O}_2$ twist fields, in the R sector.

This scheme does not say anything new about one-point functions; in particular, the functions $f^{\mathcal{O}}$ are evaluated up to normalization using analytic properties in rapidity space. We will only analyze two-point correlation functions, but the techniques and ideas introduced can be easily generalized to multi-point correlation functions.

Although our expansion (1.6) for two-point functions is of the same form as the expansion (1.5), we will find that our methods for obtaining (1.6) is clearly more powerful than an explicit evaluation of the trace as in [31], and more importantly that most of the results of [31] (where this form first appeared) are in fact incorrect¹. It is a simple matter to check that the positive verifications of the results of [31] made in the works described in the previous subsection, being only for simple fields or in particular limits, can still be made on the formulas that we propose. We will not do that explicitly here, since we have a much stronger verification of the correctness of our results through their relation with the uncontroversial form factors on the circle. Our main results can be grouped into 4 points that exhibit the differences between our approach and results, and those of previous works. They are:

1. Definition of finite-temperature form factors. The objects $f_{\epsilon_1, \dots, \epsilon_k}^{\mathcal{O}}(\theta_1, \dots, \theta_k; L)$ involved in (1.6), which we will call “finite-temperature form factors”, are *not* given by simple matrix elements $F^{\mathcal{O}}$ of \mathcal{O} in the Hilbert space on the line as was suggested in the proposal [31]. They are rather given by appropriate traces of multiple commutators $([\cdot, \cdot])$ and anti-commutators $(\{\cdot, \cdot\})$ on the Hilbert space of the Majorana theory on the line:

$$f_{\epsilon_1, \dots, \epsilon_k}^{\mathcal{O}}(\theta_1, \dots, \theta_k; L) = \langle \langle \{a^{\epsilon_1}(\theta_1), [a^{\epsilon_2}(\theta_2), \{\dots, \mathcal{O}(0, 0) \dots\}] \} \rangle \rangle_L \quad (\theta_i \neq \theta_j \ \forall i \neq j) \quad (1.7)$$

where $a^+(\theta) \equiv a^\dagger(\theta)$ is a creation operator for a particle of rapidity θ , and $a^-(\theta) \equiv a(\theta)$ is an annihilation operator for a particle of rapidity θ . Simple calculations with normal-ordered products of fermion fields, for instance, make it clear that generically, these objects are not equal to zero-temperature form factors. The difference will be explained in Point 3 below.

The techniques we use to derive such a representation are related to the physical theory of particle and hole excitations above a “thermal vacuum”, which was initially proposed more than thirty years ago and which developed into a mature theory under the name of thermo-field dynamics [40, 41] (for a review, see for instance [42]). We will explain how the traces (1.7) can be seen as matrix elements between a thermal vacuum (on the left) and a state with particle and hole excitations (on the right), and how the representation (1.6) comes from a resolution of the identity on the space of NS-sector thermal states. In fact, there are still very

¹Note that in the latter work, some singularities were neglected without good reasons (as was admitted there) in the derivation of (1.5).

few applications of the ideas of integrable quantum field theory (like the form factor program) to thermo-field dynamics (see, however, the recent study [43] of bosonization in thermo-field dynamics). Our study constitutes a step in this direction.

2. Relation between form factors on the circle and finite-temperature form factors. We will show that an appropriate analytical continuation in rapidity space of the finite-temperature form factors reproduces form factors on the circle. This links the two sets of ideas mentioned above, and, as we will see below, gives an “analytical” way of calculating form factors on the circle. The precise correspondence is given by

$${}_L\langle \tilde{n}_1, \dots, \tilde{n}_l | \mathcal{O}_L(0) | n_1, \dots, n_k \rangle_L = e^{-\frac{i\pi s}{2}} \left(\frac{2\pi}{mL} \right)^{\frac{k+l}{2}} \left(\prod_{j=1}^l \frac{1}{\sqrt{\cosh(\alpha_{\tilde{n}_j})}} \right) \left(\prod_{j=1}^k \frac{1}{\sqrt{\cosh(\alpha_{n_j})}} \right) \times \\ \times f_{+, \dots, +, -, \dots, -}^{\mathcal{O}} \left(\alpha_{n_1} + \frac{i\pi}{2}, \dots, \alpha_{n_k} + \frac{i\pi}{2}, \alpha_{\tilde{n}_l} + \frac{i\pi}{2}, \dots, \alpha_{\tilde{n}_1} + \frac{i\pi}{2}; L \right) \quad (1.8)$$

where there are k positive charges and l negative charges in the indices of $f^{\mathcal{O}}$, and where α_n are defined by

$$\sinh \alpha_n = \frac{2\pi n}{mL} \quad (n \in \mathbb{Z} + \frac{1}{2}). \quad (1.9)$$

This formula is valid for any excited states in the NS sector. When \mathcal{O} is a twist field, formula (1.8) can then be applied only if one of the bra or the ket is the vacuum, and if the branch associated to this twist field is chosen so that this vacuum is in the R sector and the excited state is in the NS sector. Using this formula, we will show that the form factor expansion on the circle and the finite-temperature form factor expansion give the same result.

3. Evaluation of finite-temperature form factors. We will evaluate explicitly the finite-temperature form factors $f_{\epsilon_1, \dots, \epsilon_k}^{\mathcal{O}}(\theta_1, \dots, \theta_k; L)$ for some fields. We find:

1. *Mixing.* For any local non-interacting field \mathcal{O} (a field that is local with respect to the fundamental fermion field), they are equal to matrix elements in the Hilbert space on the line of a sum of a finite number of local fields, including \mathcal{O} as well as operators of lower scaling dimensions and of equal or lower conformal dimensions. We will present a procedure for calculating the associated mixing matrix. This mixing can ultimately be seen as coming from the difference between the normal-ordering on the line and the normal-ordering on the circle. Note that in the proposal [31], only the energy field, among non-interacting fields, was studied, and the mixing was treated by adding to this field the identity operator in such a way that it have zero thermal expectation value. For other fields, such a simple prescription would not be enough.
2. *Leg factors.* For the order and disorder fields, they are given by their matrix elements in the Hilbert space on the line times functions depending on the individual rapidities (“leg factors”). These factors were absent in [31].

Note that, consistently with (1.8), both the mixing phenomenon and the appearance of leg factors are observed in form factors on the circle. In particular, we were able to reproduce from our results the known form factors on the circle.

We evaluate the finite-temperature form factors of order and disorder fields by deriving and solving a Riemann-Hilbert problem that completely fixes the set of finite-temperature form factors of all local twist fields (order and disorder fields and their descendants) in the Majorana field theory. The Riemann-Hilbert problem depends explicitly on the temperature. For descendant of order and disorder fields, we present a conjecture for the structure in which the mixing phenomenon can be described (without, though, calculating the associated mixing matrix).

It is important to note that since the effect of temperature on the finite-temperature form factor of order and disorder fields is only in the leg factors, a Fredholm determinant representation can straightforwardly be obtained from the finite-temperature form factor expansion, as it was done in [31]. One needs only replace the filling fraction of [31] by the filling fraction times our leg factors. However, because of these factors, following the derivation of [31] it is clear that the equation involving temperature derivatives is incorrect, in accordance with the suggestion of [17].

4. Resolution of the divergencies. Because of the poles in finite-temperature form factors of twist fields, the representation (1.6), as it stands, suffers from divergencies at colliding rapidities, when they are associated to opposite values of ϵ . Using the link, provided by (1.8), between this representation and the form factor expansion on the circle, we will describe the correct prescription that makes the integrals in (1.6) finite. This point was left slightly unclear in the proposal [31], where a similar problem arose. Such divergencies at colliding rapidities, however, were studied more carefully, in a related context, in the recent paper [28] using ideas similar to those presented here.

Some of the ideas that we will present have an easy generalization to interacting integrable models; this raises hopes that finite-temperature form factors can also be evaluated in such models. Work in this direction is in progress.

The paper is organized as follows. In Section 2, we recall the quantization of the Majorana theory on the line and on the circle. In Section 3, we introduce the space of “finite-temperature state” and explain how finite-temperature form factors give large-distance expansions of correlation functions. In Section 4, we give an argument from basic principles justifying the relation between finite-temperature form factors and form factors on the circle. In Section 5, we study finite-temperature form factors of non-interacting fields. In Section 6, we derive the Riemann-Hilbert problem for finite-temperature form factors of twist fields. In Section 7, we calculate these finite-temperature form factors. Finally, we conclude with a series of open problems.

2 Quantization of the Majorana theory on the line and on the circle

In the free Majorana theory with mass m quantized on the line, fermion operators evolved in Euclidean time τ are given by:

$$\psi(x, \tau) = \frac{1}{2} \sqrt{\frac{m}{\pi}} \int d\theta e^{\theta/2} (a(\theta) e^{ip_\theta x - E_\theta \tau} + a^\dagger(\theta) e^{-ip_\theta x + E_\theta \tau})$$

$$\bar{\psi}(x, \tau) = -\frac{i}{2} \sqrt{\frac{m}{\pi}} \int d\theta e^{-\theta/2} (a(\theta) e^{ip_\theta x - E_\theta \tau} - a^\dagger(\theta) e^{-ip_\theta x + E_\theta \tau}) \quad (2.1)$$

where the mode operators $a(\theta)$ and their Hermitian conjugate $a^\dagger(\theta)$ satisfy the canonical anti-commutation relations

$$\{a^\dagger(\theta), a(\theta')\} = \delta(\theta - \theta') \quad (2.2)$$

(other anti-commutators vanishing) and where

$$\begin{aligned} p_\theta &= m \sinh \theta , \\ E_\theta &= m \cosh \theta . \end{aligned}$$

The fermion operators satisfy the equations of motion

$$\begin{aligned} \bar{\partial}\psi(x, \tau) &\equiv \frac{1}{2} (\partial_x + i \partial_\tau) \psi = \frac{m}{2} \bar{\psi} \\ \partial\bar{\psi}(x, \tau) &\equiv \frac{1}{2} (\partial_x - i \partial_\tau) \bar{\psi} = \frac{m}{2} \psi \end{aligned} \quad (2.3)$$

and obey the equal-time anti-commutation relations

$$\{\psi(x), \psi(x')\} = \delta(x - x') , \quad \{\bar{\psi}(x), \bar{\psi}(x')\} = \delta(x - x') , \quad (2.4)$$

which is simple to derive from the representation

$$\delta(x) = \frac{1}{2\pi} \int dp e^{ipx} \quad (2.5)$$

of the delta-function. The Hilbert space \mathcal{H} is simply the Fock space over the algebra (2.2) with vacuum vector $|\text{vac}\rangle$ defined by $a(\theta)|\text{vac}\rangle = 0$. Vectors in \mathcal{H} will be denoted by

$$|\theta_1, \dots, \theta_k\rangle = a^\dagger(\theta_1) \cdots a^\dagger(\theta_k) |\text{vac}\rangle . \quad (2.6)$$

A basis is formed by taking, for instance, $\theta_1 > \cdots > \theta_k$. The Hamiltonian is given by

$$H = \int_{-\infty}^{\infty} d\theta m \cosh \theta a^\dagger(\theta) a(\theta) \quad (2.7)$$

and has the property of being bounded from below on \mathcal{H} and of generating time translations:

$$[H, \psi(x, \tau)] = \frac{\partial}{\partial \tau} \psi(x, \tau) , \quad [H, \bar{\psi}(x, \tau)] = \frac{\partial}{\partial \tau} \bar{\psi}(x, \tau) . \quad (2.8)$$

For future reference, note that the leading terms of the OPE's $\psi(x, \tau)\psi(0, 0)$ and $\bar{\psi}(x, \tau)\bar{\psi}(0, 0)$ are given by

$$\psi(x, \tau)\psi(0, 0) \sim \frac{i}{2\pi(x + i\tau)} , \quad \bar{\psi}(x, \tau)\bar{\psi}(0, 0) \sim -\frac{i}{2\pi(x - i\tau)} . \quad (2.9)$$

This normalization implies that the Majorana fermions are real, and is different from the usual normalization used in conformal field theory.

On the other hand, in the same theory quantized on the circle of circumference L , with anti-periodic conditions on the fermion fields, the fermion operators evolved in Euclidean time τ are:

$$\begin{aligned}\psi_L(x, \tau) &= \frac{1}{\sqrt{2L}} \sum_{n \in \mathbb{Z} + \frac{1}{2}} \frac{e^{\alpha_n/2}}{\sqrt{\cosh \alpha_n}} (a_n e^{ip_n x - E_n \tau} + a_n^\dagger e^{-ip_n x + E_n \tau}) \\ \bar{\psi}_L(x, \tau) &= -\frac{i}{\sqrt{2L}} \sum_{n \in \mathbb{Z} + \frac{1}{2}} \frac{e^{-\alpha_n/2}}{\sqrt{\cosh \alpha_n}} (a_n e^{ip_n x - E_n \tau} - a_n^\dagger e^{-ip_n x + E_n \tau})\end{aligned}\quad (2.10)$$

where the discrete mode operators a_n and their Hermitian conjugate a_n^\dagger satisfy the canonical anti-commutation relations

$$\{a_n^\dagger, a_{n'}\} = \delta_{n, n'} \quad (2.11)$$

(other anti-commutators vanishing) and where

$$\begin{aligned}p_n &= m \sinh \alpha_n = \frac{2\pi n}{L} \quad (n \in \mathbb{Z} + \frac{1}{2}), \\ E_n &= m \cosh \alpha_n.\end{aligned}\quad (2.12)$$

The fermion operators satisfy the equations of motion (2.3) as well as the equal-time anti-commutation relations (2.4) (with the replacement $\psi \mapsto \psi_L$ and $\bar{\psi} \mapsto \bar{\psi}_L$); the latter is simple to derive from the representation

$$\delta(x) = \frac{1}{L} \sum_{n \in \mathbb{Z} + \frac{1}{2}} e^{ip_n x} \quad (2.13)$$

of the delta-function, valid on the space of antiperiodic functions on an interval of length L . The Hilbert space \mathcal{H}_L is simply the Fock space over the algebra (2.11) with vacuum vector $|\text{vac}\rangle_L$ defined by $a_n |\text{vac}\rangle_L = 0$. Vectors in \mathcal{H}_L will be denoted by

$$|n_1, \dots, n_k\rangle_L = a_{n_1}^\dagger \cdots a_{n_k}^\dagger |\text{vac}\rangle_L. \quad (2.14)$$

A basis is formed by taking, for instance, $n_1 > \cdots > n_k$. The Hamiltonian is given by

$$H_L = \sum_{n \in \mathbb{Z} + \frac{1}{2}} m \cosh \alpha_n a_n^\dagger a_n \quad (2.15)$$

and has the property of being bounded from below on \mathcal{H}_L and of generating time translations:

$$[H_L, \psi_L(x, \tau)] = \frac{\partial}{\partial \tau} \psi_L(x, \tau), \quad [H_L, \bar{\psi}_L(x, \tau)] = \frac{\partial}{\partial \tau} \bar{\psi}_L(x, \tau). \quad (2.16)$$

The leading terms of the OPE $\psi_L(x, \tau)\psi_L(0, 0)$ and $\bar{\psi}_L(x, \tau)\bar{\psi}_L(0, 0)$ are the same as in (2.9).

3 The space of “finite-temperature states” and large-distance expansion of correlation functions

We are interested in calculating correlation functions in the theory with Hamiltonian H at finite temperature $1/L$. They are given by ratios of traces on \mathcal{H} as in (1.1). In particular, the relation (1.3) holds, where for non-interacting fields (normal-ordered products of fermion fields and of their space derivatives), the operator \mathcal{O}_L can be defined by first writing the regularization of \mathcal{O} in terms of a product of fermion operators $\psi, \bar{\psi}$ at different points (without normal ordering), then by replacing all fermion operators by their corresponding forms on the circle: $\psi \mapsto \psi_L, \bar{\psi} \mapsto \bar{\psi}_L$.

The space of “finite-temperature states” that we will use is in fact well-known in the literature: it is the Liouville space [44], or the space of operators of the theory, $\text{End}(\mathcal{H})$. As recalled in the introduction, the study of finite-temperature quantum field theory using the Liouville space goes under the name of thermo-field dynamics. It is not our intention to go into any detail of this field of study; we will recall only basic principles, in a slightly different, but equivalent, formulation, then develop certain parts for our purposes.

We will denote by \mathcal{L} the (completion of the) infinite-dimensional subspace of $\text{End}(\mathcal{H})$ spanned by products of any finite number of mode operators $a(\theta), a^\dagger(\theta)$ at different rapidities. This subspace contains local non-interacting fields of the Majorana theory. The vacuum in \mathcal{L} is defined by the identity on \mathcal{H} :

$$|\text{vac}\rangle^{\mathcal{L}} \equiv \mathbf{1}_{\mathcal{H}} , \quad (3.1)$$

and a complete basis of states is given by

$$|\theta_1, \dots, \theta_k\rangle_{\epsilon_1, \dots, \epsilon_k}^{\mathcal{L}} \equiv \prod_{j=1}^k \left(1 + e^{-\epsilon_j L E_{\theta_j}}\right) a^{\epsilon_1}(\theta_1) \cdots a^{\epsilon_k}(\theta_k) , \quad \theta_1 > \theta_2 > \cdots > \theta_k \quad (3.2)$$

where ϵ_j are signs (\pm), which will be called “charge” of the “particle” of rapidity θ_j . In fact, the sign ϵ_j is associated to the creation from the thermal bath, if it is positive, or to the absorption by the thermal bath, if it is negative, of a particle of rapidity θ_j . Note that in (3.2), we introduced the factor $\prod_{j=1}^k \left(1 + e^{-\epsilon_j L E_{\theta_j}}\right)$; this just specifies the normalization of the states, but will play an important role in the following. The inner product on \mathcal{L} is defined by

$$\langle u|v\rangle \equiv \langle\langle U^\dagger V\rangle\rangle_L , \quad \text{for } |u\rangle \equiv U, |v\rangle \equiv V . \quad (3.3)$$

In fact, as usual, it will be convenient to define states with other orderings of the rapidities by

$$|\theta_1, \dots, \theta_k\rangle_{\epsilon_1, \dots, \epsilon_k}^{\mathcal{L}} = \begin{cases} (-1)^P |\theta_{P(1)}, \dots, \theta_{P(k)}\rangle_{\epsilon_{P(1)}, \dots, \epsilon_{P(k)}}^{\mathcal{L}} & (\theta_i \neq \theta_j \forall i \neq j) \\ 0 & (\exists i \neq j \mid \theta_i = \theta_j) . \end{cases} \quad (3.4)$$

Here P is a permutation of the k first integers such that $\theta_{P(1)} > \cdots > \theta_{P(k)}$, and $(-1)^P$ is the sign of the permutation. Operators on \mathcal{H} can also be mapped to operators on \mathcal{L} by implementing their action on \mathcal{L} through their left-action or their right-action on endomorphisms of \mathcal{H} . It will be convenient for our purposes to concentrate solely on their left-action:

$$\mathcal{O} \in \text{End}(\mathcal{H}) \mapsto \mathcal{O}^{\mathcal{L}} \in \text{End}(\mathcal{L}) \equiv \text{left-action of } \mathcal{O} \text{ on } \text{End}(\mathcal{H}) . \quad (3.5)$$

Using the cyclic property of traces, it is easy to calculate the following trace:

$$\begin{aligned}
\langle\langle a(\theta)a^\dagger(\theta')\rangle\rangle_L &= e^{LE_\theta}\langle\langle a^\dagger(\theta')a(\theta)\rangle\rangle_L \\
&= e^{LE_\theta}\left(-\langle\langle a(\theta)a^\dagger(\theta')\rangle\rangle_L + \delta(\theta - \theta')\right) \\
&= \frac{\delta(\theta - \theta')}{1 + e^{-LE_\theta}}.
\end{aligned} \tag{3.6}$$

Similarly,

$$\langle\langle a^\dagger(\theta)a(\theta')\rangle\rangle_L = \frac{\delta(\theta - \theta')}{1 + e^{LE_\theta}} \tag{3.7}$$

and we also have

$$\langle\langle a(\theta)a(\theta')\rangle\rangle_L = \langle\langle a^\dagger(\theta)a^\dagger(\theta')\rangle\rangle_L = 0. \tag{3.8}$$

From these results and from using Wick's theorem (which applies for traces of products of free modes), it is easily seen that

$$\begin{aligned}
{}_{\epsilon_1, \dots, \epsilon_k} \langle\langle \theta_1, \dots, \theta_k | \theta'_1, \dots, \theta'_k \rangle\rangle_{\epsilon'_1, \dots, \epsilon'_k}^{\mathcal{L}} &= \delta(\theta_1 - \theta'_1) \cdots \delta(\theta_k - \theta'_k) \delta_{\epsilon_1, \epsilon'_1} \cdots \delta_{\epsilon_k, \epsilon'_k} \prod_{j=1}^k \left(1 + e^{-\epsilon_j LE_{\theta_j}}\right)
\end{aligned} \tag{3.9}$$

if $\theta_1 > \cdots > \theta_k$ and $\theta'_1 > \cdots > \theta'_k$, and that other inner products with such an ordering of rapidities are zero.

From these definitions, we see that finite-temperature expectation values of operators on \mathcal{H} are vacuum expectation values on \mathcal{L} :

$$\langle\langle \mathcal{O}(x, \tau) \cdots \rangle\rangle_L = {}^{\mathcal{L}}\langle \text{vac} | \mathcal{O}^{\mathcal{L}}(x, \tau) \cdots | \text{vac} \rangle^{\mathcal{L}}. \tag{3.10}$$

Hence, using the resolution of the identity on \mathcal{L} :

$$1_{\mathcal{L}} = \sum_{k=0}^{\infty} \sum_{\epsilon_1, \dots, \epsilon_k} \int \frac{d\theta_1 \cdots d\theta_k}{k! \prod_{j=1}^k (1 + e^{-\epsilon_j LE_{\theta_j}})} |\theta_1, \dots, \theta_k\rangle_{\epsilon_1, \dots, \epsilon_k}^{\mathcal{L}} {}_{\epsilon_1, \dots, \epsilon_k} \langle \theta_1, \dots, \theta_k |, \tag{3.11}$$

two-point functions can be expanded as in (1.6), where we define *finite-temperature form factors* as

$$f_{\epsilon_1, \dots, \epsilon_k}^{\mathcal{O}}(\theta_1, \dots, \theta_k; L) = {}^{\mathcal{L}}\langle \text{vac} | \mathcal{O}^{\mathcal{L}}(0, 0) | \theta_1, \dots, \theta_k \rangle_{\epsilon_1, \dots, \epsilon_k}^{\mathcal{L}}. \tag{3.12}$$

Of course, multi-point correlation functions can be expanded in similar ways. Note that in obtaining (1.6), we used the symmetries under translations (along the cylinder and around the cylinder) in order to bring out the x and τ dependence of $\mathcal{O}(x, \tau)$ in exponential factors.

It is easy to check, using the cyclic property of traces, that the finite-temperature form factors can be written in the form (1.7). This will be useful in understanding why these matrix elements are related to form factors on the circle.

It is a simple matter to observe that in the zero-temperature limit $L \rightarrow \infty$, the finite-temperature form factors reproduce the zero-temperature form factors:

$$\begin{aligned}
\lim_{L \rightarrow \infty} f_{+, \dots, +, -, \dots, -}^{\mathcal{O}}(\theta_1, \dots, \theta_{k_+}, \theta_{k_++1}, \dots, \theta_k; L) &= \langle \theta_k, \dots, \theta_{k_++1} | \mathcal{O}(0, 0) | \theta_1, \dots, \theta_{k_+} \rangle \\
&\quad (\theta_i \neq \theta_j \ \forall i \in \{1, \dots, k_+\}, j \in \{k_++1, \dots, N\})
\end{aligned} \tag{3.13}$$

where there are k_+ positive charges (+)'s in the indices of $f^{\mathcal{O}}$. For other orderings of the charges (and of the associated rapidities), we get extra minus signs. The finite-temperature form factor expansion (1.6), in the limit $L \rightarrow \infty$, then reproduces the usual form factor expansion, since only the form factors with $\epsilon_j = +$ for all j 's remain.

It is also a simple matter to observe that more general matrix elements in the space \mathcal{L} are simply related to the finite-temperature form factors (3.12):

$$f_{\epsilon_1, \dots, \epsilon_k}^{\mathcal{O}}(\theta_1, \dots, \theta_{k_+}, \theta_{k_++1}, \dots, \theta_k; L) = {}_{-\epsilon_k, \dots, -\epsilon_{k_++1}}^{\mathcal{L}} \langle \theta_k, \dots, \theta_{k_++1} | \mathcal{O}^{\mathcal{L}}(0, 0) | \theta_1, \dots, \theta_{k_+} \rangle_{\epsilon_1, \dots, \epsilon_{k_+}}^{\mathcal{L}} \\ (\theta_i \neq \theta_j \ \forall i \in \{1, \dots, k_+\}, j \in \{k_++1, \dots, k\}) . \quad (3.14)$$

Let us make two comments. First, note that expression (1.6), as it stands, is not a practical large-distance expansion. It is more convenient, as in the zero-temperature case, to analytically continue to complex values of θ_j 's, in such a way that the argument of the exponential, $i \sum_{j=1}^k \epsilon_j p_{\theta_j} x$, acquires a real part that becomes large and negative at large $|\theta_j|$ for any j . This is optimal, if $x > 0$, at $\theta_j \mapsto \theta_j + i\epsilon_j \pi/2$. However, in contrast to the zero-temperature case, the theory is not invariant under rotation, hence this shift of rapidities does not keep the form factors invariant. We will see how this shift is related to a form factor expansion on the circle.

Second, the integrals in the expansion (1.6) may suffer from singularities at colliding rapidities in their integrands. As we will see, such singularities arise in finite-temperature form factors of interacting fields when rapidity variables are associated to opposite charges. We need a prescription to define properly the finite-temperature form factor expansion in such a case. The correct prescription is the following: Consider the difference x between the positions of the first and of the second operator involved to be *positive*; then, we simply shift slightly the integration lines towards the positive or negative imaginary direction in rapidity planes, in such a way that the integration stays convergent at $\tau = 0$ and that the poles are avoided. More precisely, the integrals on $\theta_1, \dots, \theta_k$ in (1.6) should be on contours parallel to the real line with $\text{Im}(\theta_j) = \epsilon_j 0^+$. We verify that this prescription is correct, by showing independently, in Appendices A and B, the formula (1.8), and the fact that our prescription makes the finite-temperature form factor expansion (1.6) and the form factor expansion on the circle (1.4) equivalent.

A note on Ramond and Neveu-Schwartz sectors.

Because of the relation (1.2), fermion operators satisfy anti-periodic conditions in Euclidean time under the trace. The trace (1.1) with insertions of fermion fields then corresponds to correlation functions on the cylinder with anti-periodic conditions for the fermion fields around the cylinder, that is, in the NS sector in the language of the quantization on the circle.

When twist fields are inserted inside the trace, the situation is slightly different. Twist fields are end-points of branch cuts (or line defects), and on the cylinder, a semi-infinite branch cut can be in two directions. The definition of operators corresponding to twist fields is recalled in Appendix C: we denote two types of operators corresponding to any twist field \mathcal{O} by \mathcal{O}_+ and \mathcal{O}_- , with branch cut towards the positive and negative x direction along the cylinder, respectively. We call these operators “right-twist” and “left-twist” operators, respectively.

Finite-temperature correlation functions (1.1) with insertion of only one twist fields (and any

number of fermion fields) correspond then to correlation functions on the cylinder in a mixed sector, the precise mixed sector depending on which operator (right-twist or left-twist) is taken. The fermion fields have periodic conditions around the cylinder (R sector) where the branch cut stands, and anti-periodic conditions (NS sector) elsewhere. This property is essential in deriving the Riemann-Hilbert problem for finite-temperature form factors in Section 6. Correspondingly, our formula (1.6) can be used with any one of \mathcal{O}_1 and \mathcal{O}_2 a twist field, and in both cases, the branch cut can be in either direction.

Finite-temperature correlation functions with insertion of two twist fields can correspond to correlation functions on the cylinder in a pure NS sector or in a pure R sector, again depending on which operators are taken under the trace. Our formula (1.6), however, can only be used, in this case, to give an expansion of correlation functions in the R sector. This can be interpreted by the fact that in each finite-temperature matrix element of a twist field, there is no other twist field to provide another end-point of the branch cut, so that it must go to infinity along the cylinder, hence it changes the natural NS sector of the trace to an R sector at infinity. This can be seen technically by analyzing our Riemann-Hilbert problem given in Section 6, and by doing a contour-deformation analysis of the expansion (1.6), similarly to what is done in Appendix B. Correspondingly, then, the expansion (1.6) is valid, if both \mathcal{O}_1 and \mathcal{O}_2 are twist fields, only when they have branch cuts in opposite directions.

In the same spirit, note that formula (1.8) for calculating form factors on the circle from finite-temperature form factors, explained in the next section, can only be used with the vacuum (in the quantization on the circle) in the R sector, and excited states in the NS sector.

However, it can be useful to note that one could “twist” the construction of this section by defining the inner product as traces with *periodic conditions* on the fermion fields:

$$\langle u|v\rangle \equiv \langle\langle\sigma_+(-\infty,0)U^\dagger V\rangle\rangle_L/\langle\langle\sigma_+\rangle\rangle_L,$$

where the operator σ_+ is the operator associated to a primary twist field of non-zero finite-temperature average, with a branch cut on the right. The resulting finite-temperature correlation functions of fermion fields will be in the R sector. With such a definition, the formula (1.7) is still valid, but the normalization factors $\prod_{j=1}^k \left(1 + e^{-\epsilon_j L E_{\theta_j}}\right)$ introduced above should be replaced by $\prod_{j=1}^k \left(1 - e^{-\epsilon_j L E_{\theta_j}}\right)$, and a similar replacement should be made in the finite-temperature form factor expansion (1.6). Following a similar reasoning as above, the resulting expansion for the two-point function of twist fields will be valid, in this construction, only in the NS sector. The finite-temperature form factors obtained in this construction can then be used to calculate form factors on the circle with excited states in the R sector and vacuum in the NS sector, with a formula similar to (1.8). Note that starting from the form factor expansion on the circle, a representation similar to (1.6) (more precisely, after some approximations, to (1.5)) was obtained in [28] for two-point functions of order fields, with the factors $\prod_{j=1}^k \left(1 + e^{-\epsilon_j L E_{\theta_j}}\right)$ replaced by $\prod_{j=1}^k \left(1 - e^{-\epsilon_j L E_{\theta_j}}\right)$. The construction just outlined in this paragraph explains this phenomenon. In the rest of the paper, though, we will not consider this construction.

4 From finite-temperature form factors to form factors on the circle

4.1 General idea

Observe that the two equivalent expressions (1.1) and (1.3) for correlation functions at finite temperature, or on a cylinder, result from two different quantization schemes for the same theory: in the first, the equal-time slices are lines along the cylinder, whereas in the second, they are circles around it.

Observe also that the finite-temperature form factor expansion of two-point functions (1.6) could be made into a form factor expansion on the circle (1.4), if finite-temperature form factors have appropriate properties, by deforming the contours in rapidity space and taking the residues of the poles of the measure.

There are more precise observations that allow us to infer that form factors on the circle can be obtained from finite-temperature form factors. Let us develop two of them below.

First, consider a massive free theory on the Poincaré disk (the Dirac theory on the Poincaré disk was studied in [45], and we present here ideas exposed there), and consider a quantization scheme where equal-time slices are the orbits of one of its non-compact isometry (that is, the momentum operator generates isometry transformations \mathcal{I} and its spectrum is continuous). In this quantization scheme, consider the set of states $|P\rangle$ that diagonalize the momentum operator, with real continuous eigenvalues P . In fact, the states can be described by a set of momenta $\{p\}$ and a set of discrete variables (particle momenta and particle types).

Now consider matrix elements of local spinless operators in this set of states, of the form $\langle \text{vac} | \mathcal{O} | \{p\} \rangle$, and see them as analytically continued functions of momenta $\{p\}$. Choose a normalization of the states in such a way that these matrix elements are entire functions of the momenta $\{p\}$'s. With this choice, there is a measure involved in the resolution of the identity: $\mathbf{1} = \sum_k \int [dp] \rho(\{p\}) |\{p\}\rangle \langle \{p\}|$ where k is the number of particles. The singularity structure of $\rho(\{p\})$ then describes the energy spectrum of the theory in the quantization scheme where the roles of the momentum and the Hamiltonian are exchanged: the isometry \mathcal{I} is generated by the Hamiltonian. In particular, poles at purely imaginary values $p = \pm iE$ correspond to single-particle discrete eigenvalues E of the Hamiltonian in that scheme. Also, the matrix elements of \mathcal{O} in the “exchanged” quantization scheme, of the form $\langle \text{vac} | \mathcal{O} | \{E\} \rangle$ where $|\{E\}\rangle$ is an energy eigenvalue, are proportional to $\sqrt{\text{Res}(\rho(\{p\}))} \langle \text{vac} | \mathcal{O} | \{p\} \rangle$ at the values $\{p = \pm iE\}$.

The second observation is a similar one, but this time for interacting relativistic models on flat space. Consider for simplicity such a model with only one particle type (one mass). Consider a set of momentum eigenstates, parametrized by rapidity variables $\{\theta\}$. Choose a normalization of the states such that the form factors of any local field are entire functions of the total rapidity when rapidity differences are fixed, and such that the measure involved in the resolution of the identity $\mathbf{1} = \sum_k \int [d\theta] \rho(\{\theta\}) |\{\theta\}\rangle \langle \{\theta\}|$, where k is the number of particles, factorizes into a product of measures for the individual rapidities of each particle. This is the usual choice: the dependence of form factors of local fields on the total rapidity is the entire function $e^{s \sum_j \theta_j}$,

where s is the spin of the field; and the measure is $\rho(\{\theta\}) = 1$. To clarify the idea, the resolution of the identity can be written as integration over momentum variables with an appropriate relativistically invariant measure: $\mathbf{1} = \sum_k \int \left[\frac{dp}{\sqrt{p^2 + m^2}} \right] |\{\theta\}\rangle \langle \{\theta\}|$. The measure has branch cuts from $p = \pm im$ towards $\pm i\infty$: these describe, again, the energy spectrum of the theory where the roles of the momentum and of the Hamiltonian are exchanged. Of course here, by relativistic invariance, this exchange gives the same Hilbert space. It can be implemented by the shift $\theta \mapsto \theta + i\pi/2$ of rapidity variables, and the form factors are invariant, up to a phase due to the spin, under such a shift of rapidities.

Similarly, it is not hard to relate analytical continuation of special traces in angular quantization to the spectrum and matrix elements of local operators in radial quantization. Adapting these observations to the case of the cylinder, the main idea for obtaining form factors on the circle from finite-temperature form factors in the free Majorana theory can be expressed in the following (schematic) steps:

- Consider the Liouville space \mathcal{L} , where the trace

$$\frac{\text{Tr} (e^{-LH} \mathcal{O}(x, 0) \mathcal{O}(0, 0))}{\text{Tr} (e^{-LH})}$$

is a vacuum expectation value and where eigenvalues of the momentum operator are described by continuous variables, the rapidities θ_j ;

- Find a measure $\rho(\{\theta\})$ in

$$\mathbf{1}_{\mathcal{L}} = \sum_{k=0}^{\infty} \sum_{\{\epsilon\}} \int \frac{\{d\theta\}}{k!} \rho(\{\theta\}) |\{\theta\}\rangle_{\{\epsilon\}}^{\mathcal{L}} \langle \{\theta\}|_{\{\epsilon\}}^{\mathcal{L}}$$

such that matrix elements of local fields Ψ which are also local with respect to the fundamental field ψ , ${}^{\mathcal{L}}\langle \text{vac} | \Psi(0, 0) | \{\theta\} \rangle_{\{\epsilon\}}^{\mathcal{L}}$, are entire functions of the rapidities $\{\theta\}$;

- Calculate form factors on the circle by analytical continuation in the rapidity variables to the positions $\alpha_n + i\pi/2$ of the poles of the measure ρ :

$${}_L \langle \text{vac} | \Psi_L(0, 0) | \{n\} \rangle_L \propto \sqrt{\text{Res } \rho} \, {}^{\mathcal{L}} \langle \text{vac} | \Psi(0, 0) | \{\alpha_n \pm i\pi/2\} \rangle_{\{\pm\}}^{\mathcal{L}} .$$

In the last step, α_n are defined in (1.9). In the second step, the requirement that the fields Ψ be local with respect to the fundamental field ψ comes from the fact that rotation is not a symmetry on the cylinder. Finite-temperature form factors of twist fields (which are essentially the end-point of linear defects) can be expected to have a more complicated pole structure.

More than relating finite-temperature form factors to form factors on the circle, the steps above tell us that an appropriate choice of the measure is related to a simple analytical structure of finite-temperature form factors.

It is hopefully possible to generalize some of these requirements to interacting integrable models, but this will not be pursued further here.

4.2 Implementation of the general idea

For the massive Majorana theory, the first step was described in the previous section.

We now show that the second step is realized by the definition (3.12), due to the representation (1.7). Consider a complete basis $\{\Psi_j\}$ of local operators that are also local with respect to the fundamental field ψ ; these are non-interacting operators, for instance normal-ordered products of fermion operators and of their space derivatives, with coefficients that may contain integer powers of the mass m (the mass m has scaling dimension 1 and spin 0). We consider the basis $\{\Psi_j\}$ to be for the linear space of operators on the field of complex numbers without dependence on m . We can take the operators Ψ_j to have well-defined spins s_j and scaling dimensions d_j . With the fermion operators ψ and $\bar{\psi}$, they satisfy simple equal-time commutation (or anti-commutation) relations:

$$\begin{aligned} [\psi(x), \Psi_i(x')] &= \sum_{j; d_j=0, \dots, d_i-\frac{1}{2}} c_i^j \Psi_j(x') \delta^{(d_i-d_j-\frac{1}{2})}(x-x') \\ [\bar{\psi}(x), \Psi_i(x')] &= \sum_{j; d_j=0, \dots, d_i-\frac{1}{2}} \bar{c}_i^j \Psi_j(x') \delta^{(d_i-d_j-\frac{1}{2})}(x-x') . \end{aligned} \quad (4.1)$$

Here the symbol $[\cdot, \cdot]$ is the commutator if s_j is an integer or the anti-commutator if s_j is a half-integer (the spin s_j determines the statistics of the operator $\Psi_j(x')$). The sum is of course finite, because there are no fields of negative dimension. The coefficients c_i^j, \bar{c}_i^j do not depend on m . The fact that Ψ_j have well-defined spin and dimension imposes constraints on the coefficients c_i^j, \bar{c}_i^j , but we will not use these constraints here.

The operators $a(\theta)$ and $a^\dagger(\theta)$ can be decomposed in terms of the local operators $\psi(x)$ and $\bar{\psi}(x)$:

$$a^\epsilon(\theta) = \frac{1}{2} \sqrt{\frac{m}{\pi}} \int_{-\infty}^{\infty} dx e^{ip_\theta x} (e^{\theta/2} \psi(x) - \epsilon i e^{-\theta/2} \bar{\psi}(x)) . \quad (4.2)$$

The one-particle finite-temperature form factor of a field $\mathcal{O}_i(x)$ can then be written, from (1.7),

$$\begin{aligned} f_\epsilon^{\Psi_i}(\theta; L) &= \frac{1}{2} \sqrt{\frac{m}{\pi}} \int_{-\infty}^{\infty} dx e^{ip_\theta x} (e^{\theta/2} \langle [\psi(x), \Psi_i(0)] \rangle_L - \epsilon i e^{-\theta/2} \langle [\bar{\psi}(x), \Psi_i(0)] \rangle_L) \\ &= \frac{1}{2} \sqrt{\frac{m}{\pi}} \sum_{j; d_j=0}^{d_i-\frac{1}{2}} \int_{-\infty}^{\infty} dx e^{ip_\theta x} (e^{\theta/2} c_i^j - \epsilon i e^{-\theta/2} \bar{c}_i^j) \langle \Psi_j(0) \rangle_L \delta^{(d_i-d_j-\frac{1}{2})}(x) \\ &= \frac{1}{2} \sqrt{\frac{m}{\pi}} \sum_{j; d_j=0}^{d_i-\frac{1}{2}} (-\epsilon i p_\theta)^{d_i-d_j-\frac{1}{2}} (e^{\theta/2} c_i^j - \epsilon i e^{-\theta/2} \bar{c}_i^j) \langle \Psi_j(0) \rangle_L \end{aligned} \quad (4.3)$$

Note that these finite-temperature form factors are entire functions of the variable θ .

Similarly, the multi-particle form-factors are given by

$$\begin{aligned} f_{\epsilon_1, \dots, \epsilon_k}^{\Psi_{j_{k+1}}}(\theta_1, \dots, \theta_k; L) &= \\ \left(\frac{1}{2} \sqrt{\frac{m}{\pi}} \right)^k \sum_{j_1, \dots, j_k} \prod_{l=1}^k \left[(-\epsilon_l i p_{\theta_l})^{d_{j_{l+1}}-d_{j_l}-\frac{1}{2}} \left(e^{\theta_l/2} c_{j_{l+1}}^{j_l} - \epsilon_l i e^{-\theta_l/2} \bar{c}_{j_{l+1}}^{j_l} \right) \right] \langle \Psi_{j_1}(0) \rangle_L . \end{aligned} \quad (4.4)$$

Again, they are entire functions of the variables θ_l . Hence, the measure appearing in the resolution of the identity (3.11) fulfills the condition of the second step.

The third step then tells us how to evaluate form factors on the circle. The precise relation can be obtained, for instance, by calculating explicit examples for simple non-interacting fields. It is given by (1.8). A more direct derivation of this relation is presented in Appendix A. A proof that the finite-temperature form factor expansion of two-point functions (1.6) is equivalent to the expansion in form factors on the circle (1.4) is presented in Appendix B, using properties of finite-temperature form factors that are derived in the rest of this paper.

5 Finite-temperature form factors of non-interacting fields

5.1 General properties and particular cases

The expression (4.4) gives a general formula for finite-temperature form factors of local non-interacting operators. In particular, it is an easy matter to verify the property

$$f_{\epsilon_1, \dots, \epsilon_j, \dots, \epsilon_k}^{\Psi_i}(\theta_1, \dots, \theta_j + i\pi, \dots, \theta_k; L) = i f_{\epsilon_1, \dots, -\epsilon_j, \dots, \epsilon_k}^{\Psi_i}(\theta_1, \dots, \theta_j, \dots, \theta_k; L) . \quad (5.1)$$

This leads to the quasi-periodicity property

$$f_{\epsilon_1, \dots, \epsilon_j, \dots, \epsilon_k}^{\Psi_i}(\theta_1, \dots, \theta_j + 2i\pi, \dots, \theta_k; L) = -f_{\epsilon_1, \dots, \epsilon_j, \dots, \epsilon_k}^{\Psi_i}(\theta_1, \dots, \theta_j, \dots, \theta_k; L) . \quad (5.2)$$

Note that the same formula (4.4) could be used, with the replacement $\langle\langle\Psi_j\rangle\rangle_L \mapsto \langle\Psi_j\rangle$, for calculating form factors at zero temperature; indeed, the parameter L is only involved in the expectation values $\langle\langle\Psi_j\rangle\rangle_L$. This suggests that finite-temperature form factors of non-interacting operators are “not so far” from their zero-temperature limit. For instance, the finite-temperature form factors of the fundamental fermion operators are

$$f_{\pm}^{\psi}(\theta; L) = \frac{1}{2} \sqrt{\frac{m}{\pi}} e^{\theta/2} , \quad f_{\pm}^{\bar{\psi}}(\theta; L) = \mp i \frac{1}{2} \sqrt{\frac{m}{\pi}} e^{-\theta/2} , \quad (5.3)$$

which are temperature independent. On the other hand, for the energy field (from the viewpoint of the Ising field theory) $\varepsilon = i : \bar{\psi}\psi :$, the two-particle finite-temperature form factors are

$$f_{+,+}^{\varepsilon}(\theta_1, \theta_2; L) = -f_{-,-}^{\varepsilon}(\theta_1, \theta_2; L) = \frac{m}{2\pi} \sinh\left(\frac{\theta_1 - \theta_2}{2}\right) \quad (5.4)$$

and

$$f_{+,-}^{\varepsilon}(\theta_1, \theta_2; L) = -f_{-,+}^{\varepsilon}(\theta_1, \theta_2; L) = -\frac{m}{2\pi} \cosh\left(\frac{\theta_1 - \theta_2}{2}\right) \quad (5.5)$$

which agree with the zero-temperature form factors, but the thermal expectation value is

$$f^{\varepsilon}(-; L) = \langle\langle\varepsilon\rangle\rangle_L = \frac{m}{\pi} \int_0^{\infty} \frac{d\theta}{1 + e^{-mL \cosh \theta}} \quad (5.6)$$

which is non-zero and L -dependent. Those in fact are the zero-temperature form factors of the field $\varepsilon + \langle\langle\varepsilon\rangle\rangle_L \mathbf{1}$. One could re-define the field ε by subtracting this thermal expectation value, but for a more general non-interacting field, such a subtraction will not bring all its finite-temperature form factors equal to its zero-temperature form factors. This aspect was missing in [31]. In general, fields that are normal-ordered products of k fermion fields have k -particle finite-temperature form factors independent of temperature, but generically have non-zero, temperature-dependent j -particle form factors for $j < k$ (j and k being of the same parity). In order to describe this, we introduce the concept of mixing, which is a simple generalization of the description $\varepsilon \mapsto \varepsilon + \langle\langle\varepsilon\rangle\rangle_L \mathbf{1}$ of finite-temperature form factors of the energy field in terms of its zero-temperature form factors.

5.2 Mixing

Consider a complete basis $\{\Psi_a\}$ of local non-interacting operators on the field of polynomials in m – we will use indices a, b to label all elements of this set, instead of the indices i, j used in the previous section to label elements of a different basis. The elements of this set of operators can be taken to be in one to one correspondence with a basis of non-zero operators in the free massless Majorana conformal field theory. A convenient basis $\{\Psi_a\}$ is obtained by considering normal-ordered products of fermion operators ψ and $\bar{\psi}$ and of their holomorphic or anti-holomorphic derivatives, $\partial^k \psi$, $\bar{\partial}^l \bar{\psi}$, with coefficients that are independent of the mass m . We will consider their holomorphic and anti-holomorphic dimensions Δ_a , $\bar{\Delta}_a$, related as usual to their scaling dimensions $d_a = \Delta_a + \bar{\Delta}_a$ and their spins $s_a = \Delta_a - \bar{\Delta}_a$.

We will show that:

$$f_{+, \dots, +, -, \dots, -}^{\Psi_a}(\theta_1, \dots, \theta_{k_+}, \theta_{k_++1}, \dots, \theta_k; L) = \langle \theta_k, \dots, \theta_{k_++1} | \sum_b L^{d_b - d_a} M_a^b(mL) \Psi_b(0) | \theta_1, \dots, \theta_{k_+} \rangle$$

$$(\theta_i \neq \theta_j \forall i \in \{1, \dots, k_+\}, j \in \{k_++1, \dots, k\}) \quad (5.7)$$

where there are k_+ positive charges (+) in the indices of f^{Ψ_a} . Here the mixing matrix $M_a^b(mL)$ mixes operators Ψ_b of lower dimension than that of Ψ_a , $d_b < d_a$, of equal or lower associated conformal dimensions, $\Delta_b \leq \Delta_a$, $\bar{\Delta}_b \leq \bar{\Delta}_a$, and of the same statistics. The operator Ψ_a is a descendant under the fermion operator algebra of all operators Ψ_b ; in other words, the mixing occurs between Ψ_a and some of its ascendants of the same statistics. The sum over b is finite.

One can see the space \mathcal{L} as a Fock space over the algebra of modes $A_\epsilon(\theta)$ and of their Hermitian conjugate on \mathcal{L} , $A_\epsilon^\dagger(\theta)$:

$$\{A_\epsilon^\dagger(\theta), A_{\epsilon'}(\theta')\} = (1 + e^{-\epsilon LE_\theta}) \delta(\theta - \theta') \delta_{\epsilon, \epsilon'} , \quad \{A_\epsilon^\dagger(\theta), A_{\epsilon'}^\dagger(\theta')\} = \{A_\epsilon(\theta), A_{\epsilon'}(\theta')\} = 0 . \quad (5.8)$$

The vacuum is defined by $A_\epsilon(\theta)|\text{vac}\rangle^{\mathcal{L}} = 0$, and

$$|\theta_1, \dots, \theta_N\rangle_{\epsilon_1, \dots, \epsilon_N}^{\mathcal{L}} = A_{\epsilon_1}^\dagger(\theta_1) \cdots A_{\epsilon_N}^\dagger(\theta_N) |\text{vac}\rangle^{\mathcal{L}} . \quad (5.9)$$

In particular, according to (3.5), the operators $a(\theta)$ and $a^\dagger(\theta)$ act on \mathcal{L} by

$$[a^\epsilon(\theta)]^{\mathcal{L}} = \frac{A_\epsilon^\dagger(\theta)}{1 + e^{-\epsilon LE_\theta}} + \frac{A_{-\epsilon}(\theta)}{1 + e^{\epsilon LE_\theta}} . \quad (5.10)$$

One can immediately verify, for instance, that this representation on \mathcal{L} gives $\{[a^\epsilon(\theta)]^\mathcal{L}, [a^{\epsilon'}(\theta')]^\mathcal{L}\} = \delta(\theta - \theta') \delta_{\epsilon, \epsilon'}$, as it should.

Now consider an explicit set of local non-interacting operators $\{\Psi_a\}$ given by the coefficients in the expansion around $x_1 = 0, \dots, x_p = 0$ of the generating operator

$$\Psi_p(x_1, \dots, x_p) = : \psi_1(x_1, i\epsilon_1 x_1) \cdots \psi_p(x_p, i\epsilon_p x_p) : \quad (5.11)$$

where each element of the set of fields $\{\psi_j, 1 \leq j \leq p\}$ is one of ψ or $\bar{\psi}$, and where $\epsilon_j = -1$ if $\psi_j = \psi$ and $\epsilon_j = 1$ if $\psi_j = \bar{\psi}$. According to (3.5), the corresponding operator acting on \mathcal{L} by left action is denoted by $\Psi_{p,q}^\mathcal{L}(x_1, \dots, x_{p+q})$. The normal-ordering operation $: \cdot :$ on \mathcal{H} brings the operators $a(\theta)$ to the right of the operators $a^\dagger(\theta)$. This normal ordering operation can also be defined on operators acting on \mathcal{L} by bringing operators $A_+(\theta)$ and $A_-^\dagger(\theta)$ to the right of $A_+^\dagger(\theta)$ and $A_-(\theta)$, so that we can write

$$\Psi_p^\mathcal{L}(x_1, \dots, x_p) = : \psi_1^\mathcal{L}(x_1, i\epsilon_1 x_1) \cdots \psi_p^\mathcal{L}(x_p, i\epsilon_p x_p) : \quad (5.12)$$

We can also define a more natural normal-ordering operation on \mathcal{L} which brings all operators $A_+(\theta)$ and $A_-(\theta)$ to the right of $A_+^\dagger(\theta)$ and $A_-^\dagger(\theta)$; let's denote it by $: \cdot :_\mathcal{L}$. We introduce the generating operator

$$\tilde{\Psi}_p^\mathcal{L}(x_1, \dots, x_p) = : \psi_1(x_1, i\epsilon_1 x_1) \cdots \psi_p(x_p, i\epsilon_p x_p) :_\mathcal{L} = : \Psi_p^\mathcal{L}(x_1, \dots, x_p) :_\mathcal{L} \quad (5.13)$$

By Wick's theorem, we have

$$\begin{aligned} & \Psi_p^\mathcal{L}(x_1, \dots, x_p) + \sum_{m,n, m < n} (-1)^{m-n-1} C_{m,n}(x_m - x_n) \Psi_{p-2}^\mathcal{L}(x_1, \dots, \hat{x}_m, \dots, \hat{x}_n, \dots, x_p) + \dots \\ &= \\ & \tilde{\Psi}_p^\mathcal{L}(x_1, \dots, x_p) + \sum_{m,n, m < n} (-1)^{m-n-1} \tilde{C}_{m,n}(x_m - x_n) \tilde{\Psi}_{p-2}^\mathcal{L}(x_1, \dots, \hat{x}_m, \dots, \hat{x}_n, \dots, x_p) + \dots \end{aligned} \quad (5.14)$$

Here $C_{m,n}(x_m - x_n)$ is the vacuum expectation value on \mathcal{H} of the product of the fermion field at x_m and the fermion field at x_n ; that is, $C_{m,n}(x_m - x_n) = \langle \text{vac} | \psi_m(x_m, i\epsilon_m x_m) \psi_n(x_n, i\epsilon_n x_n) | \text{vac} \rangle$. Similarly, $\tilde{C}_{m,n}(x_m - x_n)$ is the vacuum expectation value on \mathcal{L} of the product of the field at x_m and the field at x_n ; that is, $\tilde{C}_{m,n}(x_m - x_n) = {}^\mathcal{L} \langle \text{vac} | \psi_m^\mathcal{L}(x_m, i\epsilon_m x_m) \psi_n^\mathcal{L}(x_n, i\epsilon_n x_n) | \text{vac} \rangle^\mathcal{L} = \langle \langle \psi_m(x_m, i\epsilon_m x_m) \psi_n(x_n, i\epsilon_n x_n) \rangle \rangle_L$. The dots (...) indicate terms with more and more contractions, until all fields are contracted, in the usual way.

By solving iteratively (5.14), we can write

$$\Psi_p^\mathcal{L}(x_1, \dots, x_p) = \tilde{\Psi}_p^\mathcal{L}(x_1, \dots, x_p) + \sum_{m,n, m < n} M_{m,n}(x_m - x_n) \tilde{\Psi}_{p-2}^\mathcal{L}(x_1, \dots, \hat{x}_m, \dots, \hat{x}_n, \dots, x_p) + \dots \quad (5.15)$$

where $M_{m,n}(x_m - x_n) = (-1)^{m-n-1} (\tilde{C}_{m,n}(x_m - x_n) - C_{m,n}(x_m - x_n))$. The dots (...) represent terms containing operators with decreasing total number of fields.

Consider the matrix element

$${}^\mathcal{L} \langle \text{vac} | \tilde{\Psi}_p^\mathcal{L}(x_1, \dots, x_p) | \theta_1, \dots, \theta_p \rangle_{\epsilon_1, \dots, \epsilon_p}^\mathcal{L} \quad (5.16)$$

Since there are as many rapidities in the state as there are fermion operators in the operator of which we take the matrix element, the normal ordering does not affect the result (there can be no internal contractions). Hence we have

$${}^{\mathcal{L}}\langle \text{vac} | \tilde{\Psi}_p^{\mathcal{L}}(x_1, \dots, x_p) | \theta_1, \dots, \theta_p \rangle_{\epsilon_1, \dots, \epsilon_p}^{\mathcal{L}} = {}^{\mathcal{L}}\langle \text{vac} | \Psi_p^{\mathcal{L}}(x_1, \dots, x_p) | \theta_1, \dots, \theta_p \rangle_{\epsilon_1, \dots, \epsilon_p}^{\mathcal{L}}. \quad (5.17)$$

Moreover, by dimensional analysis, in the expression (4.4) for the right-hand side of the equation above, only vacuum expectation values of operators of dimension 0 will remain: the identity operator. Since the finite-temperature expectation value of the identity operator is independent of the temperature, and since the only L -dependence in (4.4) occurs in expectation values, we can specialize the right-hand side of the previous equation to zero temperature ($L \rightarrow \infty$) without change in the equation. It is convenient at this point to take, without loss of generality, the p_+ first charges to be positive, and the rest to be negative, and to assume $\theta_i \neq \theta_j \forall i \in \{1, \dots, p_+\}$, $j \in \{p_+ + 1, \dots, p\}$. Then we find:

$${}^{\mathcal{L}}\langle \text{vac} | \tilde{\Psi}_p^{\mathcal{L}}(x_1, \dots, x_p) | \theta_1, \dots, \theta_{p_+}, \theta_{p_++1}, \dots, \theta_p \rangle_{+, \dots, +, -, \dots, -}^{\mathcal{L}} = \langle \theta_{p_+}, \dots, \theta_{p_++1} | \Psi_p(x_1, \dots, x_p) | \theta_1, \dots, \theta_{p_+} \rangle. \quad (5.18)$$

Now, all matrix elements of $\tilde{\Psi}_p^{\mathcal{L}}(x_1, \dots, x_p)$ between the vacuum and states containing more or less than p rapidities are zero because of the finite-temperature normal ordering. Similarly, all zero-temperature matrix elements of $\Psi_p(x_1, \dots, x_p)$ between states whose total number of particle is more or less than p are zero. Hence we can write in general:

$${}^{\mathcal{L}}\langle \text{vac} | \tilde{\Psi}_p^{\mathcal{L}}(x_1, \dots, x_p) | \theta_1, \dots, \theta_{k_+}, \theta_{k_++1}, \dots, \theta_k \rangle_{+, \dots, +, -, \dots, -}^{\mathcal{L}} = \langle \theta_k, \dots, \theta_{k_++1} | \Psi_p(x_1, \dots, x_p) | \theta_1, \dots, \theta_{k_+} \rangle \quad (5.19)$$

for any k and any k_+ . Using these equalities and the relation (5.15), we have

$$\begin{aligned} f_{+, \dots, +, -, \dots, -}^{\Psi_p(x_1, \dots, x_p)}(\theta_1, \dots, \theta_{k_+}, \theta_{k_++1}, \dots, \theta_k; L) = \\ \langle \theta_k, \dots, \theta_{k_++1} | \Psi_p(x_1, \dots, x_p) | \theta_1, \dots, \theta_{k_+} \rangle + \\ \sum_{m < n} M_{m,n}(x_m - x_n) \langle \theta_k, \dots, \theta_{k_++1} | \Psi_{p-2}(x_1, \dots, \hat{x}_m, \dots, \hat{x}_n, \dots, x_p) | \theta_1, \dots, \theta_{k_+} \rangle + \\ \dots \end{aligned} \quad (5.20)$$

This is exactly of the form (5.7), once expanded in powers of x_m 's. From this calculation, we see that the elements of the mixing matrix can be seen as coming from the difference between the normal-ordering $:\cdot:_{\mathcal{L}}$, which for local fields on \mathcal{H} is essentially a normal-ordering on the circle, and the usual normal-ordering on the line $:\cdot:$.

It is possible to describe the mixing matrix in a more precise fashion. Recall that the space \mathcal{L} contains local non-interacting fields. The mixing matrix can be seen as an operator acting on \mathcal{L} , such that a state in \mathcal{L} corresponding to the field $\Psi_a(0)$ is mapped to the state corresponding to $\sum_b L^{d_b - d_a} M_a^b(mL) \Psi_b(0)$.

Consider the map $\Psi(0) \mapsto \tilde{\Psi}(0)$ that maps any normal-ordered non-interacting operator $\Psi(0) \in \text{End}(\mathcal{H})$ to the operator $\tilde{\Psi}(0) \in \text{End}(\mathcal{H})$ such that $\tilde{\Psi}^{\mathcal{L}}(0) = : \Psi^{\mathcal{L}}(0) :_{\mathcal{L}}$ (the finite-temperature

form factors of $\tilde{\Psi}(0)$ are equal to the zero-temperature form factors of $\Psi(0)$, as described above). We saw above how to construct $\tilde{\Psi}(0)$: it indeed exists, and it is unique. Now consider the state $|\Psi\rangle^{\mathcal{L}} = \Psi^{\mathcal{L}}(0)|\text{vac}\rangle^{\mathcal{L}}$ as well as the state $|\tilde{\Psi}\rangle^{\mathcal{L}} = \tilde{\Psi}^{\mathcal{L}}(0)|\text{vac}\rangle^{\mathcal{L}}$ in \mathcal{L} . The mixing matrix $M_a^b(mL)$ is defined by

$$|\Psi_a\rangle^{\mathcal{L}} = \sum_b L^{d_b-d_a} M_a^b(mL) |\tilde{\Psi}_b\rangle^{\mathcal{L}} . \quad (5.21)$$

Define the operator $U \in \text{End}(\mathcal{L})$ generating this mixing matrix:

$$|\Psi_a\rangle^{\mathcal{L}} = U |\tilde{\Psi}_a\rangle^{\mathcal{L}} . \quad (5.22)$$

From the arguments above, it can be shown that

$$U = \exp \left[\int_{-\infty}^{\infty} d\theta \frac{A_-(\theta) A_+(\theta)}{1 + e^{LE_\theta}} \right] . \quad (5.23)$$

6 Riemann-Hilbert problem for finite-temperature form factors

6.1 Non-interacting fields

In Section 5, we described finite-temperature form factors of non-interacting fields. They are all given by entire functions of the rapidity variables θ_i 's, as are zero-temperature form factors of non-interacting fields; the main non-trivial phenomenon is that of mixing between a field and its ascendants.

In fact one can verify that the linear space of functions f^Ψ of the variables θ_i , $i = 1, \dots, k$ with the properties:

1. f^Ψ acquires a sign under exchange of any two of the rapidity variables;
2. f^Ψ has the quasi-periodicity property (5.2);
3. f^Ψ is an entire functions of all its variables;

reproduces the linear space of k -particle finite-temperature form factors $f_{+, \dots, +}^\Psi(\theta_1, \dots, \theta_k; L)$ of non-interacting operators Ψ . This constitutes a very simple Riemann-Hilbert problem. Of course, the solution to this Riemann-Hilbert problem also reproduces the set of zero-temperature form factors; in order to disentangle the finite-temperature form factors, one needs the additional information about the mixing matrix.

6.2 Riemann-Hilbert problem associated to right-twist operators

As recalled in Appendix C, there are two types of operators associated to each twist field: those with branch cut on their right ("right-twist operators"), and those with branch cut on their left

(“left-twist operators”). As explained in Section 3, with an appropriate choice of the direction of the branch cuts, one can obtain two-point functions of twist fields in the NS sector and in the R sector. In this subsection, we will consider right-twist operators only.

One can expect a description for the finite-temperature form factors of twist fields (which are “interacting” fields) in the same spirit as the one in the previous sub-section for non-interacting fields.

Consider the function

$$f(\theta_1, \dots, \theta_k; L) = f_{+, \dots, +}^{\mathcal{O}_+}(\theta_1, \dots, \theta_k; L)$$

where \mathcal{O}_+ is the operator with branch cut on its right representing a twist field: this can be the order field σ_+ or the disorder field μ_+ , or any of their conformal descendants (that is, fields which reproduce conformal descendants in the massless limit). Conformal descendants include space derivatives, as well as other fields related to action of higher conformal Virasoro modes on twist fields. A way of describing such descendants is by taking the limit $x \rightarrow 0$ of the finite part of the OPE $\Psi(x)\sigma_+(0)$ or $\Psi(x)\mu_+(0)$, where Ψ is any bosonic non-interacting field.

The function f solves the following Riemann-Hilbert problem:

1. Statistics of free particles: f acquires a sign under exchange of any two of the rapidity variables;
2. Quasi-periodicity:

$$f(\theta_1, \dots, \theta_j + 2i\pi, \dots, \theta_k; L) = -f(\theta_1, \dots, \theta_j, \dots, \theta_k; L), \quad j = 1, \dots, k;$$

3. Analytic structure: f is analytic as function of all of its variables θ_j , $j = 1, \dots, k$ everywhere on the complex plane except at simple poles. In the region $\text{Im}(\theta_j) \in [-i\pi, i\pi]$, $j = 1, \dots, k$, its analytic structure is specified as follows:

- (a) Thermal poles and zeroes: $f(\theta_1, \dots, \theta_k; L)$ has poles at

$$\theta_j = \alpha_n - \frac{i\pi}{2}, \quad n \in \mathbb{Z}, \quad j = 1, \dots, k$$

and has zeroes at

$$\theta_j = \alpha_n - \frac{i\pi}{2}, \quad n \in \mathbb{Z} + \frac{1}{2}, \quad j = 1, \dots, k,$$

- (b) Kinematical poles: $f(\theta_1, \dots, \theta_k; L)$ has poles, as a function of θ_k , at $\theta_j \pm i\pi$, $j = 1, \dots, k-1$ with residues given by

$$f(\theta_1, \dots, \theta_k; L) \sim \pm \frac{(-1)^{k-j}}{\pi} \frac{1 + e^{-LE\theta_j}}{1 - e^{-LE\theta_j}} \frac{f(\theta_1, \dots, \hat{\theta}_j, \dots, \theta_{k-1}; L)}{\theta_k - \theta_j \mp i\pi}.$$

In order to have other finite-temperature form factors than those with all positive charges, one more relation needs to be used. We have:

4. Crossing symmetry:

$$f_{\epsilon_1, \dots, \epsilon_j, \dots, \epsilon_k}^{\mathcal{O}+}(\theta_1, \dots, \theta_j + i\pi, \dots, \theta_k; L) = i f_{\epsilon_1, \dots, -\epsilon_j, \dots, \epsilon_k}^{\mathcal{O}+}(\theta_1, \dots, \theta_j, \dots, \theta_k; L) .$$

The same relation is also valid for non-interacting fields; recall (5.1). The name “crossing symmetry” is inspired by the zero-temperature case. To make it more obvious, define the functions

$$f(\theta'_1, \dots, \theta'_l | \theta_1, \dots, \theta_k; L) = {}_{+, \dots, +} \langle \theta'_1, \dots, \theta'_l | \mathcal{O}_+^{\mathcal{L}}(0) | \theta_1, \dots, \theta_k \rangle_{+, \dots, +}^{\mathcal{L}} . \quad (6.1)$$

These are in fact distributions, and can be decomposed in terms supported at separated rapidities $\theta'_i \neq \theta_j$, $\forall i, j$, and terms supported at colliding rapidities, $\theta'_i = \theta_j$ for some i and j . We will denote the former by $f^{sep.}(\theta'_1, \dots, \theta'_l | \theta_1, \dots, \theta_k; L)$, and the latter by $f^{coll.}(\theta'_1, \dots, \theta'_l | \theta_1, \dots, \theta_k; L)$. Under integration over rapidity variables, the former gives principal value integrals. Recalling the property (3.14), we have

$$f^{sep.}(\theta'_1, \dots, \theta'_l | \theta_1, \dots, \theta_k; L) = f_{+, \dots, +, -, \dots, -}^{\mathcal{O}+}(\theta_1, \dots, \theta_k, \theta'_1, \dots, \theta'_l; L)$$

for $(\theta'_i \neq \theta_j \ \forall i \in \{1, \dots, l\}, j \in \{1, \dots, k\})$, where on the right-hand side, there are k positive charges (+), and l negative charges (-). Analytically extending from its support the distribution $f^{sep.}$ to a function of complex rapidity variables, crossing symmetry can then be written

$$\begin{aligned} f^{sep.}(\theta'_1, \dots, \theta'_l | \theta_1, \dots, \theta_k + i\pi; L) &= i f^{sep.}(\theta'_1, \dots, \theta'_l, \theta_k | \theta_1, \dots, \theta_{k-1}; L) , \\ f^{sep.}(\theta'_1, \dots, \theta'_l + i\pi | \theta_1, \dots, \theta_k; L) &= i f^{sep.}(\theta'_1, \dots, \theta'_{l-1} | \theta_1, \dots, \theta_k, \theta'_l; L) , \end{aligned}$$

which justifies its name.

It is worth mentioning that the distributive terms corresponding to colliding rapidities satisfy a set of recursive equations:

5. Colliding part of matrix elements:

$$\begin{aligned} f^{coll.}(\theta'_1, \dots, \theta'_l | \theta_1, \dots, \theta_k; L) &= \\ \sum_{i=1}^l \sum_{j=1}^k (-1)^{l+k-i-j} \frac{1 + e^{-LE_{\theta_j}}}{1 - e^{LE_{\theta_j}}} \delta(\theta'_i - \theta_j) f(\theta'_1, \dots, \hat{\theta}'_i, \dots, \theta'_l | \theta_1, \dots, \hat{\theta}_j, \dots, \theta_k; L) . \end{aligned}$$

Note that the colliding part vanishes in the limit of zero temperature, $L \rightarrow \infty$. Finally, it is instructive to re-write the distribution $f(\theta'_1, \dots, \theta'_l | \theta_1, \dots, \theta_k; L)$ as an analytical function with slightly shifted rapidities, plus a distribution, using the relation

$$\frac{1}{\theta - i0^+} = i\pi\delta(\theta) + \underline{\mathbb{P}}\left(\frac{1}{\theta}\right) \quad (6.2)$$

where $\underline{\mathbb{P}}$ means that we must take the principal value integral under integration. Defining the disconnected part $f^{disconn.}(\theta'_1, \dots, \theta'_l | \theta_1, \dots, \theta_k; L)$ of the matrix element (6.1) as

$$f(\theta'_1, \dots, \theta'_l | \theta_1, \dots, \theta_k; L) = f^{sep.}(\theta'_1 - i0^+, \dots, \theta'_l - i0^+ | \theta_1, \dots, \theta_k; L) + f^{disconn.}(\theta'_1, \dots, \theta'_l | \theta_1, \dots, \theta_k; L) \quad (6.3)$$

where again we analytically extend from its support the distribution $f^{sep.}$ to a function of complex rapidity variables, we find that the disconnected part satisfies the recursion relations

$$f^{disconn.}(\theta'_1, \dots, \theta'_l | \theta_1, \dots, \theta_k; L) = \sum_{i=1}^l \sum_{j=1}^k (-1)^{l+k-i-j} (1 + e^{-LE_{\theta_j}}) \delta(\theta'_i - \theta_j) f(\theta'_1, \dots, \hat{\theta}'_i, \dots, \theta'_l | \theta_1, \dots, \hat{\theta}_j, \dots, \theta_k; L) .$$

Note that the factor $(1 + e^{-LE_{\theta_j}}) \delta(\theta'_i - \theta_j)$ appearing inside the double sum is just the overlap $\mathcal{L}_+(\theta'_i | \theta_j) \mathcal{L}_+$, so that the equation above can be naturally represented as a sum of disconnected diagrams.

We will derive below all of the three points in the Riemann-Hilbert problem above, as well as crossing symmetry and the recursive formula for the colliding part of matrix elements.

6.3 Mixing for twist fields

It is natural to suppose that, like in the case of non-interacting fields, some “mixing” occurs between a twist field \mathcal{O} and its ascendants – fields of lower dimension and with the same locality index and statistics as those of \mathcal{O} – in calculating the finite-temperature form factors. We describe here a conjecture for the way mixing should occur. We will use right-twist operators for this description, but we expect that the same mixing should occur amongst left-twist operators.

First note that the fact that finite-temperature form factors have exponential asymptotic behavior, when all rapidity variables are sent to positive or negative infinity simultaneously, can be seen as following from the requirement of convergence of the form factor expansion (1.6) in the region $0 < \tau < L$. Consider a solution $\tilde{f}_{\tilde{\mathcal{O}}_+}$ to the Riemann-Hilbert problem above (Points 1, 2 and 3), with the property that

$$\tilde{f}_{\tilde{\mathcal{O}}_+}(\theta_1 + \beta, \dots, \theta_k + \beta; L) \sim \langle \text{vac} | \tilde{\mathcal{O}}_+(0) | \theta_1, \dots, \theta_k \rangle e^{s\beta} \quad \text{as } \beta \rightarrow \pm\infty$$

for some twist operators $\tilde{\mathcal{O}}_+$ of spins s . It will be clear in the next section that such solutions exist. Consider a mixing matrix for twist fields whose elements $M_{\mathcal{O}_+}^{\tilde{\mathcal{O}}_+}(mL)$ are parametrized by two operators \mathcal{O}_+ and $\tilde{\mathcal{O}}_+$. This mixing matrix is characteristic of the theory, and its elements are non-zero only for operators $\tilde{\mathcal{O}}_+$ that are ascendants of \mathcal{O}_+ . The finite-temperature form factors of \mathcal{O}_+ are the following linear combinations:

$$f(\theta_1, \dots, \theta_k; L) = \sum_{\tilde{\mathcal{O}}_+} M_{\mathcal{O}_+}^{\tilde{\mathcal{O}}_+}(mL) \tilde{f}_{\tilde{\mathcal{O}}_+}(\theta_1, \dots, \theta_k; L) . \quad (6.4)$$

We do not yet have a full derivation of this, neither a description of the mixing matrix, but it seems a natural assumption.

6.4 Derivation of the Riemann-Hilbert problem associated to right-twist operators

Point 1.

The first point of the Riemann-Hilbert problem of sub-section 6.2 is a direct consequence of the definition of the finite-temperature form factors and of the canonical anti-commutation relations of the free fermionic modes.

Point 3a.

We now derive the position of the residues and zeroes stated in Point 3a. We first concentrate on one-particle finite-temperature form factors. Consider the two-point functions

$$g(x, \tau) = {}^{\mathcal{L}}\langle \text{vac} | \mathcal{O}_+^{\mathcal{L}}(0, 0) \psi^{\mathcal{L}}(x, \tau) | \text{vac} \rangle^{\mathcal{L}}. \quad (6.5)$$

and

$$\tilde{g}(x, \tau) = {}^{\mathcal{L}}\langle \text{vac} | \psi^{\mathcal{L}}(x, \tau) \mathcal{O}_+^{\mathcal{L}}(0, 0) | \text{vac} \rangle^{\mathcal{L}}. \quad (6.6)$$

Their finite-temperature form factor expansions are, from (5.3),

$$g(x, \tau) = \frac{1}{2} \sqrt{\frac{m}{\pi}} \int d\theta e^{\theta/2} \left(\frac{f_+^{\mathcal{O}_+}(\theta; L)}{1 + e^{-LE_\theta}} e^{-ip_\theta x + E_\theta \tau} + \frac{f_-^{\mathcal{O}_+}(\theta; L)}{1 + e^{LE_\theta}} e^{ip_\theta x - E_\theta \tau} \right) \quad (6.7)$$

and

$$\tilde{g}(x, \tau) = \frac{1}{2} \sqrt{\frac{m}{\pi}} \int d\theta e^{\theta/2} \left(\frac{f_-^{\mathcal{O}_+}(\theta; L)}{1 + e^{-LE_\theta}} e^{ip_\theta x - E_\theta \tau} + \frac{f_+^{\mathcal{O}_+}(\theta; L)}{1 + e^{LE_\theta}} e^{-ip_\theta x + E_\theta \tau} \right). \quad (6.8)$$

The one-particle form factors are essentially fixed, up to normalization, by the requirements:

- convergence of the form factor expansion of $g(x, \tau)$ (6.7) in the region $-L < \tau < 0$;
- quasi-periodicity of the analytic continuation in τ : $g(x, \tau + L) = -g(x, \tau)$ if $x < 0$ and $g(x, \tau + L) = g(x, \tau)$ if $x > 0$.

For $\tilde{g}(x, \tau)$, the first requirement is replaced by a convergence in $0 < \tau < L$, and the second requirement holds unchanged; they also essentially fix the one-particle form factors. The first requirement implies that the form factors $f_{\pm}^{\mathcal{O}_+}(\theta; L)$ must go as $\propto e^{p\theta}$ for some number p as $\theta \rightarrow \infty$, and as $\propto e^{q\theta}$ for some number q as $\theta \rightarrow -\infty$. The second specifies the positions of poles and zeros.

The second requirement will be satisfied if we can deform the contour in θ -space and take poles at appropriate values. With the generalization to multi-particle finite-temperature form factors in mind, we will consider these deformations in $g(x, \tau)$ for $x < 0$ only and in $\tilde{g}(x, \tau)$ for $x > 0$ only. For $x < 0$, in (6.7), we can shift the θ -contour upwards for the term containing $e^{-ip_\theta x}$ and downwards for the term containing $e^{ip_\theta x}$. Consider the functions

$$g_+(\theta) = \frac{f_+^{\mathcal{O}_+}(\theta; L)}{1 + e^{-LE_\theta}}, \quad g_-(\theta) = \frac{f_-^{\mathcal{O}_+}(\theta; L)}{1 + e^{LE_\theta}}.$$

The anti-periodicity condition at $x < 0$ will be satisfied if: the function $g_+(\theta)$ has poles at $\theta = \alpha_n + i\pi/2$ (recall (1.9)) for $n \in \mathbb{Z} + \frac{1}{2}$; the function $g_-(\theta)$ has poles at $\theta = \alpha_n - i\pi/2$ for $n \in \mathbb{Z} + \frac{1}{2}$; and $e^{i\pi/4}g_+(\theta' + i\pi/2) + e^{-i\pi/4}g_-(\theta' - i\pi/2) = 0$ for all θ' real (except at the positions of the poles). Then, by deforming the contours, we get the sum of $2\pi i$ times the residues of the poles of $g_+(\theta)$ (equivalently, $-2\pi i$ times the residues of the poles of $g_-(\theta)$) and the resulting function is anti-periodic in τ . For $x > 0$, consider (6.8). Define

$$\tilde{g}_+(\theta) = \frac{f_+^{\mathcal{O}_+}(\theta; L)}{1 + e^{LE_\theta}} , \quad \tilde{g}_-(\theta) = \frac{f_-^{\mathcal{O}_+}(\theta; L)}{1 + e^{-LE_\theta}} .$$

With similar arguments for the function $\tilde{g}(x, \tau)$ to be periodic: the function $\tilde{g}_+(\theta)$ has poles at $\theta = \alpha_n - i\pi/2$ for $n \in \mathbb{Z}$; the function $g_-(\theta)$ has poles at $\theta = \alpha_n + i\pi/2$ for $n \in \mathbb{Z}$; and $e^{-i\pi/4}g_+(\theta' - i\pi/2) + e^{i\pi/4}g_-(\theta' + i\pi/2) = 0$ for all θ' real (except at the positions of the poles). There must be no other poles in $\text{Im}(\theta) \in [-\pi/2, \pi/2]$ for the four functions $g_\pm(\theta)$ and $\tilde{g}_\pm(\theta)$ (note that $g_\pm(\theta)/\tilde{g}_\pm(\theta)$ are entire functions of θ). Below we will argue that there are no other poles in the wider range $\text{Im}(\theta) \in [-\pi, \pi]$. Assuming further no other types of singularities than simple poles, we conclude that $g_\pm(\theta)$ and $\tilde{g}_\pm(\theta)$ must have no other singularities than those mentioned above in the region $\text{Im}(\theta) \in [-\pi, \pi]$.

Recall, from the general relation (3.14), that $f_-^{\mathcal{O}_+}(\theta; L) = \left(f_+^{\mathcal{O}_+}(\theta; L)\right)^\dagger$. For the function $f_+^{\mathcal{O}_+}(\theta; L)$, this gives the following conditions:

- $f_+^{\mathcal{O}_+}(\theta; L)$ has poles at $\theta = \alpha_n - \frac{i\pi}{2}$, $n \in \mathbb{Z}$ and has zeroes at $\theta = \alpha_n - \frac{i\pi}{2}$, $n \in \mathbb{Z} + \frac{1}{2}$;
- $f_+^{\mathcal{O}_+}(\theta; L)$ does not have poles for $\text{Im}(\theta) \in [-\pi, \pi]$ except for those mentioned above;
- $\text{Re}\left(e^{-\frac{i\pi}{4}}f_+^{\mathcal{O}_+}(\theta' - i\pi/2; L)\right) = 0$, $\text{Re}\left(e^{\frac{i\pi}{4}}f_+^{\mathcal{O}_+}(\theta' + i\pi/2; L)\right) = 0$ for $\theta' \in \mathbb{R}$.

The first and second points are the one-particle case of Point 3a. We will verify, in explicit calculations of finite-temperature form factors below, that the last point is in fact a consequence of the full Riemann-Hilbert problem of sub-section (6.2) along with (3.14).

The generalization to multi-particle finite-temperature form factors $f_{\epsilon_1, \dots, \epsilon_k}^{\mathcal{O}_+}(\theta_1, \dots, \theta_k; L)$ goes along the same lines. We need to calculate the multi-point function with k insertions of the fermion field ψ at k different points. We need to consider two cases: one with these insertions on the right of \mathcal{O}_+ , and one with these insertions on its left. We only have to assume that there are no poles in $|\text{Im}(\theta_i - \theta_j)| \leq \pi/2$ if the two rapidities θ_i and θ_j are associated to the same charge, and in $|\text{Im}(\theta_i - \theta_j)| \leq \pi$, except possibly at $\theta_i = \theta_j$, if the two rapidities are associated to opposite charges. The poles at $\theta_i = \theta_j$ (below, we will show that they are present) are taken care of by the prescription stated at the end of Section 3. This prescription does not affect our arguments, since we were careful to shift rapidity contours in directions making the two-point function convergent at $\tau = 0$ with positive difference between the positions of the first and of the second operator. Then, repeating the arguments, we find Point 3a.

Points 2 and 4.

Point 2 is evidently a consequence of Point 4. Hence we will show the latter.

We concentrate again, first, on the one-particle form factors. Consider the function $g(x, \tau)$ (6.5) and its form factor expansion (6.7). Consider $x < 0$, and shift the contour in the first term as $\theta \mapsto \theta + i\pi/2$ and in the second term as $\theta \mapsto \theta - i\pi/2$. This gives, as explained above, a sum of residues. The same sum of residues can be obtained, up to an overall minus sign, by taking the θ -contour slightly above the line of imaginary part $\pi/2$ in the first term, and slightly below the line of imaginary part $-\pi/2$ in the second term. This gives again a representation of the two-point function valid in the region $-L < \tau < 0$, $x < 0$. From this representation, we can shift the θ -contours all the way to the line of imaginary part π in the first term, and to the line of imaginary part $-\pi$ in the second term, taking residues of poles, if any. We obtain a representation of exactly the same form as (6.7), valid in the same regions of x and τ . Since a representation of this form is unique, each coefficient of the exponentials should be the same; in particular, there should be no poles in the region $\pi/2 < \text{Im}(\theta) \leq \pi$ for the first term and $-\pi < \text{Im}(\theta) \leq -\pi/2$ for the second. The same argument can be applied to the representation (6.8) of the function $\tilde{g}(x, \tau)$ in the region $x > 0$. The fact that the representation is unique, taking into account the prefactor $e^{\theta/2}$ in (6.7) and (6.8) and the minus sign occurring when going beyond the line of poles at imaginary parts $\pm\pi/2$, gives

$$f_{\pm}^{\mathcal{O}^+}(\theta + i\pi; L) = i f_{\mp}^{\mathcal{O}^+}(\theta; L) ,$$

which is the one-particle case of Point 4.

The generalization to multi-particle form factors goes along the same lines. The prescription stated at the end of Section 3, for the case of poles appearing at colliding rapidity variables, stays invariant under shifting by $\pm i\pi$ the rapidity integration lines in the argument above.

Points 3b and 5.

We now prove Points 3b and 5. We first concentrate on the two-particle case. Consider the distribution

$$h(\theta_2|\theta_1) \equiv \langle \langle a(\theta_2) \mathcal{O}_+(0) a^\dagger(\theta_1) \rangle \rangle_L = \frac{f(\theta_2|\theta_1; L)}{(1 + e^{-LE_{\theta_1}})(1 + e^{-LE_{\theta_2}})} \quad (6.9)$$

where on the right-hand side we use the definition (6.1). Expanding the modes in fermion fields using (4.2), we have

$$\begin{aligned} h(\theta_2|\theta_1) = & \frac{m}{4\pi} \int dx_2 dx_1 e^{-ip_2 x_2 + ip_1 x_1} \times \\ & \times \left[e^{\frac{\theta_2 + \theta_1}{2}} \langle \langle \psi(x_2) \mathcal{O}_+(0) \psi(x_1) \rangle \rangle_L - i e^{\frac{\theta_2 - \theta_1}{2}} \langle \langle \psi(x_2) \mathcal{O}_+(0) \bar{\psi}(x_1) \rangle \rangle_L \right. \\ & \left. + i e^{-\frac{\theta_2 - \theta_1}{2}} \langle \langle \bar{\psi}(x_2) \mathcal{O}_+(0) \psi(x_1) \rangle \rangle_L + e^{-\frac{\theta_2 + \theta_1}{2}} \langle \langle \bar{\psi}(x_2) \mathcal{O}_+(0) \bar{\psi}(x_1) \rangle \rangle_L \right] . \end{aligned}$$

Consider the first term inside the bracket. We want the main behavior of the expression around $\theta_2 = \theta_1$. It is obtained by taking x_2 and x_1 very large, both with the same sign. In this limit, the correlation functions factorize, so the main contribution of the first term is contained into

$$\frac{m}{4\pi} \int dx_2 dx_1 e^{-ip_2 x_2 + ip_1 x_1} e^{\frac{\theta_2 + \theta_1}{2}} \text{sign}(-x_1) \langle \langle \psi(x_2) \psi(x_1) \rangle \rangle_L^{\text{sign}(x_1)} \langle \langle \mathcal{O}_+(0) \rangle \rangle_L \quad (6.10)$$

where the correlation function of fermion fields is taken with *periodic* conditions around the cylinder if $x_1 > 0$, and anti-periodic conditions if $x_1 < 0$:

$$\langle\langle \cdots \rangle\rangle_L^\pm \quad : \quad \text{trace with periodic (+) / anti-periodic (-) conditions on the fermion fields .}$$

The factor $\text{sign}(-x_1)$ arises because we must put the operator $\psi(x_2)$ at a slightly positive Euclidean time $\tau_2 = 0^+$ and $\psi(x_1)$ at a slightly negative Euclidean time $\tau_1 = 0^-$ in the initial correlation function, and we must take $\psi(x_1)$ through the branch cut produced by $\mathcal{O}_+(0)$, if $x_1 > 0$, before we can factorize. Changing variables, this is

$$\frac{m}{4\pi} \int dx_2 dx_1 e^{-ip_2 x_2 + i(p_1 - p_2)x_1} e^{\frac{\theta_2 + \theta_1}{2}} \text{sign}(-x_1) \langle\langle \psi(x_2) \psi(0) \rangle\rangle_L^{\text{sign}(x_1)} \langle\langle \mathcal{O}_+(0) \rangle\rangle_L . \quad (6.11)$$

The integral over x_1 is a sum of two distributions, one supported on $p_2 - p_1 \neq 0$, one supported at $p_2 - p_1 = 0$. They can be evaluated by using the distributional equations

$$\int dx e^{ipx} \text{sign}(x) = 2i \underline{\text{P}} \left(\frac{1}{p} \right) , \quad \int dx e^{ipx} = 2\pi \delta(p) \quad (6.12)$$

where $\underline{\text{P}}$ means principal value. The part supported on $p_2 - p_1 \neq 0$ is

$$\frac{im}{4\pi} \underline{\text{P}} \left(\frac{1}{p_2 - p_1} \right) \int dx_2 e^{-ip_2 x_2} e^{\frac{\theta_2 + \theta_1}{2}} [\langle\langle \psi(x_2) \psi(0) \rangle\rangle_L^+ + \langle\langle \psi(x_2) \psi(0) \rangle\rangle_L^-] \langle\langle \mathcal{O}_+(0) \rangle\rangle_L \quad (6.13)$$

whereas the part supported at $p_2 - p_1 = 0$ is

$$\frac{m}{4} \delta(p_2 - p_1) \int dx_2 e^{-ip_2 x_2} e^{\frac{\theta_2 + \theta_1}{2}} [\langle\langle \psi(x_2) \psi(0) \rangle\rangle_L^- - \langle\langle \psi(x_2) \psi(0) \rangle\rangle_L^+] \langle\langle \mathcal{O}_+(0) \rangle\rangle_L . \quad (6.14)$$

Putting all terms together, the former gives

$$h^{sep.}(\theta_2|\theta_1) \sim \frac{im}{4\pi} \underline{\text{P}} \left(\frac{1}{\theta_2 - \theta_1} \right) \frac{1}{E_{\theta_1}} \int dx_2 e^{-ip_1 x_2} [\langle\langle \Psi(\theta_1|x_2) \rangle\rangle_L^+ + \langle\langle \Psi(\theta_1|x_2) \rangle\rangle_L^-] \langle\langle \mathcal{O}_+(0) \rangle\rangle_L \quad (6.15)$$

and the latter gives

$$h^{coll.}(\theta_2|\theta_1) = \frac{m}{4} \delta(\theta_2 - \theta_1) \frac{1}{E_{\theta_1}} \int dx_2 e^{-ip_1 x_2} [\langle\langle \Psi(\theta_1|x_2) \rangle\rangle_L^- - \langle\langle \Psi(\theta_1|x_2) \rangle\rangle_L^+] \langle\langle \mathcal{O}_+(0) \rangle\rangle_L \quad (6.16)$$

where

$$\Psi(\theta_1|x_2) = e^{\theta_1} \psi(x_2) \psi(0) - i \psi(x_2) \bar{\psi}(0) + i \bar{\psi}(x_2) \psi(0) + e^{-\theta_1} \bar{\psi}(x_2) \bar{\psi}(0) .$$

In order to evaluate the integral over x_2 , consider the traces

$$\begin{aligned} \langle\langle a(\theta_2) a^\dagger(\theta_1) \rangle\rangle_L^- &= \frac{\delta(\theta_2 - \theta_1)}{1 + e^{-LE_{\theta_1}}} \\ \langle\langle a(\theta_2) a^\dagger(\theta_1) \rangle\rangle_L^+ &= \frac{\delta(\theta_2 - \theta_1)}{1 - e^{-LE_{\theta_1}}} . \end{aligned}$$

A derivation similar to the one above but now applied to these objects gives

$$\frac{m}{2} \delta(p_2 - p_1) \int dx_2 e^{-ip_1 x_2} \langle\langle \Psi(\theta_1|x_2) \rangle\rangle_L^\pm = \frac{\delta(\theta_2 - \theta_1)}{1 \mp e^{-LE_{\theta_1}}} . \quad (6.17)$$

Hence,

$$\begin{aligned}
h^{sep.}(\theta_2|\theta_1) &\sim \frac{i}{2\pi} \mathbb{P}\left(\frac{1}{\theta_2 - \theta_1}\right) \left[\frac{1}{1 - e^{-LE_{\theta_1}}} + \frac{1}{1 + e^{-LE_{\theta_1}}} \right] \langle\langle \mathcal{O}_+(0) \rangle\rangle_L \\
&= \frac{i}{\pi} \mathbb{P}\left(\frac{1}{\theta_2 - \theta_1}\right) \frac{1}{(1 - e^{-LE_{\theta_1}})(1 + e^{-LE_{\theta_1}})} \langle\langle \mathcal{O}_+(0) \rangle\rangle_L
\end{aligned} \tag{6.18}$$

from which we have

$$f^{sep.}(\theta_2|\theta_1; L) \sim \frac{i}{\pi} \frac{1 + e^{-LE_{\theta_1}}}{1 - e^{-LE_{\theta_1}}} \mathbb{P}\left(\frac{1}{\theta_2 - \theta_1}\right) \langle\langle \mathcal{O}_+(0) \rangle\rangle_L. \tag{6.19}$$

Combined with crossing symmetry, this proves Point 3b for the case $k = 2, j = 1$. On the other hand, the delta-function part is given by

$$\begin{aligned}
h^{coll.}(\theta_2|\theta_1) &= \frac{1}{2} \delta(\theta_2 - \theta_1) \left[\frac{1}{1 + e^{-LE_{\theta_1}}} - \frac{1}{1 - e^{-LE_{\theta_1}}} \right] \langle\langle \mathcal{O}_+(0) \rangle\rangle_L \\
&= \frac{\delta(\theta_2 - \theta_1)}{(1 - e^{LE_{\theta_1}})(1 + e^{-LE_{\theta_1}})} \langle\langle \mathcal{O}_+(0) \rangle\rangle_L
\end{aligned} \tag{6.20}$$

which shows Point 5 for $l = k = 1, i = j = 1$.

A similar argument holds for $k > 2$, with extra minus signs coming from the odd statistics of the fermion fields, of their modes, and of the twist operator if it has non-zero form factors with odd particle numbers only.

6.5 Riemann-Hilbert problem associated to left-twist operators

Similar arguments can be used to analyze finite-temperature form factors of twist fields with branch cut on their left. We only state here the corresponding Riemann-Hilbert problem.

Consider the function

$$f(\theta_1, \dots, \theta_k; L) = f_{+, \dots, +}^{\mathcal{O}_-}(\theta_1, \dots, \theta_k; L)$$

where \mathcal{O}_- is the operator with branch cut on its left representing a twist field.

The function f solves the following Riemann-Hilbert problem:

1. Statistics of free particles: f acquires a sign under exchange of any two of the rapidity variables;
2. Quasi-periodicity:

$$f(\theta_1, \dots, \theta_j + 2i\pi, \dots, \theta_k; L) = -f(\theta_1, \dots, \theta_j, \dots, \theta_k; L), \quad j = 1, \dots, k;$$

3. Analytic structure: f is analytic as function of all of its variables $\theta_j, j = 1, \dots, k$ everywhere on the complex plane except at simple poles. In the region $\text{Im}(\theta_j) \in [-i\pi, i\pi], j = 1, \dots, k$, its analytic structure is specified as follows:

(a) Thermal poles and zeroes: $f(\theta_1, \dots, \theta_k; L)$ has poles at

$$\theta_j = \alpha_n + \frac{i\pi}{2}, \quad n \in \mathbb{Z}, \quad j = 1, \dots, k$$

and has zeroes at

$$\theta_j = \alpha_n + \frac{i\pi}{2}, \quad n \in \mathbb{Z} + \frac{1}{2}, \quad j = 1, \dots, k,$$

(b) Kinematical poles: $f(\theta_1, \dots, \theta_k; L)$ has poles, as a function of θ_k , at $\theta_j \pm i\pi$, $j = 1, \dots, k-1$ with residues given by

$$f(\theta_1, \dots, \theta_k; L) \sim \mp \frac{(-1)^{k-j}}{\pi} \frac{1 + e^{-LE_{\theta_j}}}{1 - e^{-LE_{\theta_j}}} \frac{f(\theta_1, \dots, \hat{\theta}_j, \dots, \theta_{k-1}; L)}{\theta_k - \theta_j \mp i\pi}.$$

Again, in order to have other finite-temperature form factors than those with all positive charges, one more relation needs be used. We have:

4. Crossing symmetry:

$$f_{\epsilon_1, \dots, \epsilon_j, \dots, \epsilon_k}^{\mathcal{O}_-}(\theta_1, \dots, \theta_j + i\pi, \dots, \theta_k; L) = i f_{\epsilon_1, \dots, -\epsilon_j, \dots, \epsilon_k}^{\mathcal{O}_-}(\theta_1, \dots, \theta_j, \dots, \theta_k; L).$$

Moreover, matrix elements

$$f(\theta'_1, \dots, \theta'_l | \theta_1, \dots, \theta_k; L) = {}_{+, \dots, +}^{\mathcal{L}} \langle \theta'_1, \dots, \theta'_l | \mathcal{O}_-^{\mathcal{L}}(0) | \theta_1, \dots, \theta_k \rangle_{+, \dots, +}^{\mathcal{L}}$$

can again be decomposed in terms supported at separated rapidities $\theta'_i \neq \theta_j$, $\forall i, j$ (which give principal value integrals under integration), and terms supported at colliding rapidities, $\theta'_i = \theta_j$ for some i and j , denoted respectively by $f^{sep.}(\theta'_1, \dots, \theta'_l | \theta_1, \dots, \theta_k; L)$ and $f^{coll.}(\theta'_1, \dots, \theta'_l | \theta_1, \dots, \theta_k; L)$. Recalling the property (3.14), we have

$$f^{sep.}(\theta'_1, \dots, \theta'_l | \theta_1, \dots, \theta_k; L) = f_{+, \dots, +, -, \dots, -}^{\mathcal{O}_-}(\theta_1, \dots, \theta_k, \theta'_1, \dots, \theta'_l; L)$$

for $(\theta'_i \neq \theta_j \forall i \in \{1, \dots, l\}, j \in \{1, \dots, k\})$, where on the right-hand side, there are k positive charges (+), and l negative charges (-). The distributive terms corresponding to colliding rapidities satisfy the same set of recursive equations as in the case of right-twist operators:

5. Colliding part of matrix elements:

$$f^{coll.}(\theta'_1, \dots, \theta'_l | \theta_1, \dots, \theta_k; L) = \sum_{i=1}^l \sum_{j=1}^k (-1)^{l+k-i-j} \frac{1 + e^{-LE_{\theta_j}}}{1 - e^{LE_{\theta_j}}} \delta(\theta'_i - \theta_j) f(\theta'_1, \dots, \hat{\theta}'_i, \dots, \theta'_l | \theta_1, \dots, \hat{\theta}_j, \dots, \theta_k; L).$$

Finally, we can again re-write the distribution $f(\theta'_1, \dots, \theta'_l | \theta_1, \dots, \theta_k; L)$ as an analytical function with slightly shifted rapidities, plus a distribution, using this time the relation

$$\frac{1}{\theta + i0^+} = -i\pi\delta(\theta) + \underline{\mathbb{P}}\left(\frac{1}{\theta}\right). \quad (6.21)$$

Defining the disconnected part $f^{disconn.}(\theta'_1, \dots, \theta'_l | \theta_1, \dots, \theta_k; L)$ of the matrix element (6.1) as

$$f(\theta'_1, \dots, \theta'_l | \theta_1, \dots, \theta_k; L) = f^{sep.}(\theta'_1 + i0^+, \dots, \theta'_l + i0^+ | \theta_1, \dots, \theta_k; L) + f^{disconn.}(\theta'_1, \dots, \theta'_l | \theta_1, \dots, \theta_k; L) \quad (6.22)$$

where again we analytically extend from its support the distribution $f^{sep.}$ to a function of complex rapidity variables, we find that the disconnected part satisfies the recursion relations

$$f^{disconn.}(\theta'_1, \dots, \theta'_l | \theta_1, \dots, \theta_k; L) = \sum_{i=1}^l \sum_{j=1}^k (-1)^{l+k-i-j} (1 + e^{-LE_{\theta_j}}) \delta(\theta'_i - \theta_j) f(\theta'_1, \dots, \hat{\theta}'_i, \dots, \theta'_l | \theta_1, \dots, \hat{\theta}_j, \dots, \theta_k; L).$$

7 Finite-temperature form factors of twist fields

For the order and disorder fields, σ and μ (again, see Appendix C for the definition of the associated operators σ_{\pm} and μ_{\pm}), the solutions to the Riemann-Hilbert problem of sub-section 6.2 are completely fixed (up to a normalization) by the asymptotic behavior $\sim O(1)$ at $\theta \rightarrow \pm\infty$, since they are primary fields of spin 0. Note that the method of computing one-particle finite-temperature form factors by solving the Riemann-Hilbert problem with this asymptotic behavior is very similar to the method used by Fonseca and Zamolodchikov [27] for calculating form factors on the circle.

For the one-particle finite-temperature form factor of the disorder operator with a branch cut on its right, the solution is

$$f_{\pm}^{\mu+}(\theta; L) = e^{\pm \frac{i\pi}{4}} C(L) \exp \left[\mp \int_{-\infty \mp i0^+}^{\infty \mp i0^+} \frac{d\theta'}{2\pi i} \frac{1}{\sinh(\theta - \theta')} \ln \left(\frac{1 + e^{-LE_{\theta'}}}{1 - e^{-LE_{\theta'}}} \right) \right] \quad (7.1)$$

for some real constant $C(L)$. This is in agreement with the Hermiticity of μ_+ , which gives $(f_{\pm}^{\mu+}(\theta; L))^* = f_{\mp}^{\mu+}(\theta; L)$ for θ real. Using

$$\frac{1}{\sinh(\theta - (\theta' \pm i0^+))} = \pm i\pi\delta(\theta - \theta') + \underline{\mathbb{P}}\left(\frac{1}{\sinh(\theta - \theta')}\right).$$

this can also be written

$$f_{\pm}^{\mu+}(\theta; L) = C(L) e^{\pm \frac{i\pi}{4}} \sqrt{\frac{1 + e^{-LE_{\theta}}}{1 - e^{-LE_{\theta}}}} \exp \left[\mp \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi i} \underline{\mathbb{P}}\left(\frac{1}{\sinh(\theta - \theta')}\right) \ln \left(\frac{1 + e^{-LE_{\theta'}}}{1 - e^{-LE_{\theta'}}} \right) \right]. \quad (7.2)$$

That this is a solution can be checked by verifying the asymptotic behavior $f_{\pm}^{\mu+}(\theta; L) \sim e^{\pm \frac{i\pi}{4}} C(L)$ as $|\theta| \rightarrow \infty$, and by verifying that the functions $f_{\pm}^{\mu+}(\theta; L)$ have poles and zeros at the proper positions. Positions of poles and zeros are the values of θ such that when analytically continued from real values, a pole at $\sinh(\theta - \theta') = 0$ in the integrand of (7.1) and one of the logarithmic branch points pinch the θ' contour of integration. The fact that these positions correspond to poles and zeros can be deduced most easily from the functional relation

$$f_{\pm}^{\mu+}(\theta; L) f_{\pm}^{\mu+}(\theta \pm i\pi; L) = \pm i C(L)^2 \frac{1 + e^{-LE_{\theta}}}{1 - e^{-LE_{\theta}}} . \quad (7.3)$$

Note that this implies the quasi-periodicity property

$$f_{\pm}^{\mu+}(\theta + 2i\pi; L) = -f_{\pm}^{\mu+}(\theta; L) . \quad (7.4)$$

It is also easy to see that the crossing symmetry relation is satisfied. Also, since $C(L)$ is real, one can check the validity of the relation $\text{Re} \left(e^{-\frac{i\pi}{4}} f_{+}^{\mathcal{O}+}(\theta' - i\pi/2; L) \right) = 0$, $\text{Re} \left(e^{\frac{i\pi}{4}} f_{+}^{\mathcal{O}+}(\theta' + i\pi/2; L) \right) = 0$ for θ' real; this relation was seen as a consequence of general principles in the proof of Point 3a in sub-section 6.4; it is now seen as a consequence of the Riemann-Hilbert problem along with Hermiticity.

For the operator μ_{-} with a branch cut on its left, one can check similarly that the function

$$f_{\pm}^{\mu-}(\theta; L) = f_{\pm}^{\mu+}(\theta - i\pi; L) = -i f_{\mp}^{\mu+}(\theta; L) \quad (7.5)$$

solves the Riemann-Hilbert problem of sub-section 6.5. Explicitly,

$$f_{\pm}^{\mu-}(\theta; L) = -ie^{\mp \frac{i\pi}{4}} C(L) \exp \left[\pm \int_{-\infty \pm i0^{+}}^{\infty \pm i0^{+}} \frac{d\theta'}{2\pi i \sinh(\theta - \theta')} \ln \left(\frac{1 + e^{-LE_{\theta'}}}{1 - e^{-LE_{\theta'}}} \right) \right] . \quad (7.6)$$

In particular, we observe that $(f_{\pm}^{\mu-}(\theta; L))^* = -f_{\mp}^{\mu-}(\theta; L)$, which is in agreement with the anti-Hermiticity of the operator μ_{-} (see Appendix C). Note that we chose the same constant $C(L)$ as a normalization for both $f_{\pm}^{\mu-}$ and $f_{\pm}^{\mu+}$. This is not a consequence of the Riemann-Hilbert problem, but can be checked by explicitly calculating the normalization. The normalization is calculated in Appendix D, and is given by

$$C(L) = \frac{\langle \langle \sigma \rangle \rangle_L}{\sqrt{2\pi}} \quad (7.7)$$

where the average $\langle \langle \sigma \rangle \rangle_L$ was calculated in [29] (the average at zero-temperature (that is, $L \rightarrow \infty$) can be found in [46]) and is given by

$$m^{\frac{1}{8}} 2^{\frac{1}{12}} e^{-\frac{1}{8}} A^{\frac{3}{2}} \exp \left[\frac{(mL)^2}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\theta_1 d\theta_2}{(2\pi)^2} \frac{\sinh \theta_1 \sinh \theta_2}{\sinh(mL \cosh \theta_1) \sinh(mL \cosh \theta_2)} \ln \left| \left(\coth \frac{\theta_1 - \theta_2}{2} \right) \right| \right]$$

where A is Glaisher's constant.

Multi-particle finite-temperature form factors can be easily constructed from the well-known zero-temperature form factors (first calculated in [8]), by adjoining “leg factors”, which are just normalized one-particle finite-temperature form factors:

$$f_{+, \dots, +}^{\mathcal{O}+}(\theta_1, \dots, \theta_k; L) = i^{[\frac{k}{2}]} \langle \langle \sigma \rangle \rangle_L \left(\prod_{j=1}^k \frac{f_{+}^{\mu+}(\theta_j; L)}{\langle \langle \sigma \rangle \rangle_L} \right) \prod_{1 \leq i < j \leq k} \tanh \left(\frac{\theta_j - \theta_i}{2} \right) \quad (7.8)$$

where \mathcal{O}_+ is σ_+ if k is even, and μ_+ if k is odd. The symbol $[k/2]$ equals the greatest integer smaller than or equal to $k/2$. This satisfies the condition on thermal poles and zeroes simply from the properties of the leg factors, and it can be verified that this satisfies the quasi-periodicity condition and the kinematical pole condition, Point 2 and Point 3b of sub-section 6.2, respectively. Using crossing symmetry, Point 4, it is a simple matter to obtain the formula for other values of the charges:

$$f_{\epsilon_1, \dots, \epsilon_k}^{\mathcal{O}_+}(\theta_1, \dots, \theta_k; L) = i^{[k/2]} \langle\langle \sigma \rangle\rangle_L \left(\prod_{j=1}^k \frac{f_{\epsilon_j}^{\mu_+}(\theta_j; L)}{\langle\langle \sigma \rangle\rangle_L} \right) \prod_{1 \leq i < j \leq k} \left(\tanh \left(\frac{\theta_j - \theta_i}{2} \right) \right)^{\epsilon_i \epsilon_j}. \quad (7.9)$$

Similarly, we have

$$f_{\epsilon_1, \dots, \epsilon_k}^{\mathcal{O}_-}(\theta_1, \dots, \theta_k; L) = i^{[k/2]} \langle\langle \sigma \rangle\rangle_L \left(\prod_{j=1}^k \frac{f_{\epsilon_j}^{\mu_-}(\theta_j; L)}{\langle\langle \sigma \rangle\rangle_L} \right) \prod_{1 \leq i < j \leq k} \left(\tanh \left(\frac{\theta_j - \theta_i}{2} \right) \right)^{\epsilon_i \epsilon_j} \quad (7.10)$$

where \mathcal{O}_- is σ_- if k is even, and μ_- if k is odd.

It is easy to check, using (1.8), that the formulas above for finite-temperature form factors reproduce the known form factors on the circle [25, 26, 27].

Also, it is a simple matter to obtain a Fredholm determinant representation for the two-point function of twist fields in the R sector. This is derived in Appendix E.

The zero-temperature form factors of descendant twist fields in the Majorana theory were analyzed in [47], as solutions to the form factor equations. It was found that the set of form factors of descendant fields of a given spin s is described by multiplying form factors of primary twist fields by symmetric polynomials in e^{θ_j} , $j = 1, \dots, k$ (where θ_j are the rapidities of the form factor) homogeneous of degree s . Similarly, we expect that finite-temperature form factors of descendant twist fields are obtained by multiplying those of primary twist fields by symmetric polynomials. It is easy to check that the requirements of the Riemann-Hilbert problems of Sections 6.2 and 6.5 are still satisfied after such an operation. However, as explained in the previous section, since rotation is no longer a symmetry on the cylinder, we cannot identify the spin of a descendant with the homogeneous degree of symmetric polynomials. Instead, we must rely on the conjecture in sub-section 6.3 in order to describe how to combine appropriate symmetric polynomials. Unfortunately, we have not yet calculated the mixing matrix for twist fields, hence we cannot derive more explicit formulas here for descendants.

8 Conclusions

We have derived a method for writing large-distance expansions of finite-temperature correlation functions. For this purpose, we introduced the space of operators \mathcal{L} on which finite-temperature quantum averages are vacuum expectation values, and we obtained the expansion of two-point functions by inserting a resolution of the identity between the two operators. We defined finite-temperature form factors, which are appropriately normalized matrix elements on

\mathcal{L} of local operators, and we described at length their properties. They are related, by analytical continuation in rapidity space, to form factors in the quantization scheme on the circle. Any finite-temperature form factor of a non-interacting field can be seen as the zero-temperature form factor of a sum of non-interacting fields, including itself and fields of lower dimension. The coefficients in this sum are independent of the matrix element evaluated. We described this phenomenon by constructing the associated mixing matrix; note that this was overlooked in the initial approach [31]. We derived the Riemann-Hilbert problem that defines the set of all finite-temperature form factors of twist fields, and calculated finite-temperature form factors of order and disorder fields.

There are still many open questions:

We have only implicitly described how to obtain large-distance expansion of correlation functions with more than two operators: one should insert the resolution of the identity between every pairs of adjacent operators. The resulting expression involves matrix elements where both vectors in \mathcal{L} correspond to excited states. For non-interacting fields this does not cause any difficulties, but for twist fields, one must be more careful. We evaluated such matrix elements of twist fields, as distributions with terms supported at colliding rapidities (which vanish at zero temperature) and others supported at separated rapidities. Using these distributions, all integrals over rapidity variables are well-defined, and it would be nice to explicitly observe the agreement of the resulting expression for multi-point correlation functions with a form factor expansion on the circle. In particular, from the results presented here, it should be possible to calculate matrix elements of twist fields in the quantization on the circle where both vectors correspond to excited states.

We have described how to obtain form factors on the circle from finite-temperature form factors only for excited states in the NS sector, where fermion fields have anti-periodic condition around the cylinder. It would be interesting to describe matrix elements in a similar way with excited states in the R sector. As we mentioned, this can be done by defining the inner product on \mathcal{L} as a trace with a twist operator at, say, position $x = -\infty$ and having a branch cut on its right.

We have explained how to obtain the mixing matrix associated to non-interacting fields. In particular, we have obtained an operator U that describes the action of this matrix on the non-interacting fields seen as vectors in \mathcal{L} . The existence of such an operator is well-known in conformal field theory: there, the space of operators is isomorphic to the Hilbert space in radial quantization (which is isomorphic to the Hilbert space in any quantization on closed lines), and the mixing operator is described by an operator on the Hilbert space performing a transformation to the cylinder. It would be interesting to relate more explicitly the operator U that we wrote with a transformation to the cylinder.

We obtain finite-temperature form factors of order and disorder fields. It is important to note that form factors for the right- and left-operators both specialize to the zero-temperature form factors only in the region $|\text{Im}(\theta)| < \pi/2$ of the complex plane of rapidity variables θ . In connection to this, observe that the quasi-periodicity relation does not agree with the one satisfied by zero-temperature form factors. At finite temperature, this quasi-periodicity relation does not seem to contain any information about the mutual locality of the twist field with respect to the fundamental fermion fields. This semi-locality is included in the thermal poles

and zeroes and in the particular form of the kinematical residue. It would be interesting to generalize this to other semi-locality index, by deriving the Riemann-Hilbert problem in the free Dirac theory for scaling twist fields associated to the $U(1)$ symmetry (in this connection, see the recent results [48]). But the thermal poles and zeroes might not be sole consequences of semi-locality; they might be interpreted as coming from self-interaction of the field around the cylinder as well, hence could be present also for fields with zero semi-locality index in interacting theories.

It could be interesting to obtain the non-linear differential equations describing the two-point function of twist fields from the Fredholm determinant representation resulting from the finite-temperature form factor expansion. This would modify some of the results of [31]; in particular, it is easy to see that the equation evolving temperature derivatives would not hold due to the non-trivial leg factors, and it is not clear if there is an equation replacing it.

Finally, probably the most important future development from the ideas presented here is the definition and calculation of finite-temperature form factors in interacting integrable models, and the identification of the associated measure of integration in the resolution of the identity. An important new idea presented here is the relation between finite-temperature form factors, seen as matrix elements on the Hilbert space of operators, and form factors on the circle; in particular, the relation between the measure of integration and the spectrum on the circle. If, for fields that are mutually local with respect to fundamental fields, the finite-temperature form factors equations are the same as the well-known zero-temperature form factor equations, as was assumed in previous approaches, then the only non-trivial element is the measure of integration in the resolution of the identity. This measure was indeed subject of controversy [36, 37, 35]. Our ideas give a guideline for defining this measure of integration. Note, however, that in view of the works [21, 22], the rapidity variables might turn out not be the most appropriate ones for defining finite-temperature form factors.

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A Derivation of the relation (1.8)

A general argument for formula (1.8) can be obtained from the relation between traces and expectation values on the circle, (1.1) and (1.3). Recall the expression of mode operators in terms of fermion operators (4.2). From this, we can write

$$\begin{aligned}
& {}_{+, \dots, +} \langle \tilde{\theta}_1, \dots, \tilde{\theta}_l | \mathcal{O}^{\mathcal{L}}(0, 0) | \theta_1, \dots, \theta_k \rangle_{+, \dots, +}^{\mathcal{L}} \\
&= \langle \langle a(\tilde{\theta}_l) \cdots a(\tilde{\theta}_1) \mathcal{O}(0, 0) a^\dagger(\theta_1) \cdots a^\dagger(\theta_k) \rangle \rangle_L \left(\prod_{j=1}^l (1 + e^{-LE_{\tilde{\theta}_j}}) \right) \left(\prod_{j=1}^k (1 + e^{-LE_{\theta_j}}) \right)
\end{aligned}$$

$$\begin{aligned}
&= \left(\frac{1}{2} \sqrt{\frac{m}{\pi}} \right)^k \int dx_1 \cdots dx_l dx_1 \cdots dx_k e^{-i \sum_{j=1}^l p_{\theta_j} x_j + i \sum_{j=1}^k p_{\theta_j} x_j} \times \\
&\quad \times \langle \left(e^{\frac{\tilde{\theta}_l}{2}} \psi(x_l) + i e^{-\frac{\tilde{\theta}_l}{2}} \bar{\psi}(x_l) \right) \cdots \left(e^{\frac{\tilde{\theta}_1}{2}} \psi(x_1) + i e^{-\frac{\tilde{\theta}_1}{2}} \bar{\psi}(x_1) \right) \mathcal{O}(0,0) \times \\
&\quad \times \left(e^{\frac{\theta_1}{2}} \psi(x_1) - i e^{-\frac{\theta_1}{2}} \bar{\psi}(x_1) \right) \cdots \left(e^{\frac{\theta_k}{2}} \psi(x_k) - i e^{-\frac{\theta_k}{2}} \bar{\psi}(x_k) \right) \rangle_L \times \\
&\quad \times \left(\prod_{j=1}^l \left(1 + e^{-LE_{\tilde{\theta}_j}} \right) \right) \left(\prod_{j=1}^k \left(1 + e^{-LE_{\theta_j}} \right) \right) \\
&= e^{\frac{i\pi s}{2}} \left(\frac{1}{2} \sqrt{\frac{m}{\pi}} \right)^k \int dx_1 \cdots dx_l dx_1 \cdots dx_k e^{-i \sum_{j=1}^l p_{\theta_j} x_j + i \sum_{j=1}^k p_{\theta_j} x_j} \times \\
&\quad \times {}_L \langle \left(e^{\frac{\tilde{\theta}_l}{2} + \frac{i\pi}{4}} \psi_L(x_l) + i e^{-\frac{\tilde{\theta}_l}{2} - \frac{i\pi}{4}} \bar{\psi}_L(x_l) \right) \cdots \left(e^{\frac{\tilde{\theta}_1}{2} + \frac{i\pi}{4}} \psi_L(x_1) + i e^{-\frac{\tilde{\theta}_1}{2} - \frac{i\pi}{4}} \bar{\psi}_L(x_1) \right) \mathcal{O}_L(0,0) \times \\
&\quad \times \left(e^{\frac{\theta_1}{2} + \frac{i\pi}{4}} \psi_L(x_1) - i e^{-\frac{\theta_1}{2} - \frac{i\pi}{4}} \bar{\psi}_L(x_1) \right) \cdots \left(e^{\frac{\theta_k}{2} + \frac{i\pi}{4}} \psi_L(x_k) - i e^{-\frac{\theta_k}{2} - \frac{i\pi}{4}} \bar{\psi}_L(x_k) \right) \rangle_L \times \\
&\quad \times \left(\prod_{j=1}^l \left(1 + e^{-LE_{\tilde{\theta}_j}} \right) \right) \left(\prod_{j=1}^k \left(1 + e^{-LE_{\theta_j}} \right) \right) \\
&= e^{\frac{i\pi s}{2}} \left(\frac{1}{2} \sqrt{\frac{m}{\pi}} \right)^k {}_L \langle \text{vac} | w_-(\tilde{\theta}_l) \cdots w_-(\tilde{\theta}_1) \mathcal{O}_L(0,0) w_+(\theta_1) \cdots w_+(\theta_k) | \text{vac} \rangle_L \tag{A.1}
\end{aligned}$$

where we define the operators

$$w_\epsilon(\theta) \equiv (1 + e^{-LE_\theta}) \int dx e^{i\epsilon p_\theta x} \left(e^{\frac{\theta}{2} + \frac{i\pi}{4}} \psi_L(0, x) - i\epsilon e^{-\frac{\theta}{2} - \frac{i\pi}{4}} \bar{\psi}_L(0, x) \right). \tag{A.2}$$

We are interested in the analytical continuation $\theta \mapsto \theta + i\pi/2$ of the matrix element (A.1) for all rapidity variables, then in taking the limit where $\theta \rightarrow \alpha_n$ (1.9) of this analytical continuation. The result of this limit is obtained by taking the analytical continuation $W_\pm(\theta) = w_\pm(\theta + i\pi/2)$ of the operators above, then by taking, in all operators $W_+(\theta)$ on the right of $\mathcal{O}(0)$, only the negative part of the integral over x , and in all operators $W_-(\theta)$ on the left of $\mathcal{O}(0)$, only the positive part of the integral over x . Indeed, the variable x is a time variable in the quantization on the circle, positive x corresponding to positive time, and the result of the limit is obtained by looking at the time-ordered part of the correlation function.

Hence we have

$$\begin{aligned}
W_+(\theta) &\sim (1 + e^{-iLp_\theta}) \int_{-\infty}^0 dx e^{-E_\theta x} \left[e^{\frac{\theta}{2} + \frac{i\pi}{2}} \psi_L(0, x) - i e^{-\frac{\theta}{2} - \frac{i\pi}{2}} \bar{\psi}_L(0, x) \right] \\
&= (1 + e^{-iLp_\theta}) \frac{i}{\sqrt{2L}} \sum_{n \in \mathbb{Z} + \frac{1}{2}} \int_{-\infty}^0 dx \frac{e^{-E_\theta x}}{\sqrt{\cosh(\alpha_n)}} \times \\
&\quad \times \left[e^{-E_n x} \left(e^{\frac{\theta + \alpha_n}{2}} + e^{-\frac{\theta + \alpha_n}{2}} \right) a_n + e^{E_n x} \left(e^{\frac{\theta + \alpha_n}{2}} - e^{-\frac{\theta + \alpha_n}{2}} \right) a_n^\dagger \right] \\
&= (1 + e^{-iLp_\theta}) \frac{i}{\sqrt{2L}} \sum_{n \in \mathbb{Z} + \frac{1}{2}} \frac{1}{\sqrt{\cosh(\alpha_n)}} \left[-\frac{e^{\frac{\theta + \alpha_n}{2}} + e^{-\frac{\theta + \alpha_n}{2}}}{E_n + E_\theta} a_n + \frac{e^{\frac{\theta + \alpha_n}{2}} - e^{-\frac{\theta + \alpha_n}{2}}}{E_n - E_\theta} a_n^\dagger \right].
\end{aligned}$$

In the last step, we used analytical continuation in the exponents in order to perform the integral. We can now extract the result of the limit $\theta \rightarrow \alpha_n$:

$$\begin{aligned} W_+(\theta) &\sim \frac{i}{\sqrt{2L}} \frac{2 \sinh(\alpha_n)}{\sqrt{\cosh(\alpha_n)}} \frac{1 + e^{-iLp\theta}}{E_n - E_\theta} a_n^\dagger \\ &\sim \sqrt{2L \cosh(\alpha_n)} a_n^\dagger. \end{aligned} \quad (\text{A.3})$$

On the other hand,

$$\begin{aligned} W_-(\theta) &\sim (1 + e^{-iLp\theta}) \int_0^\infty dx e^{E_\theta x} \left[e^{\frac{\theta}{2} + \frac{i\pi}{2}} \psi_L(0, x) + i e^{-\frac{\theta}{2} - \frac{i\pi}{2}} \bar{\psi}_L(0, x) \right] \\ &= (1 + e^{-iLp\theta}) \frac{i}{\sqrt{2L}} \sum_{n \in \mathbb{Z} + \frac{1}{2}} \int_0^\infty dx \frac{e^{E_\theta x}}{\sqrt{\cosh(\alpha_n)}} \times \\ &\quad \times \left[e^{-E_n x} \left(e^{\frac{\theta + \alpha_n}{2}} - e^{-\frac{\theta + \alpha_n}{2}} \right) a_n + e^{E_n x} \left(e^{\frac{\theta + \alpha_n}{2}} + e^{-\frac{\theta + \alpha_n}{2}} \right) a_n^\dagger \right] \\ &= (1 + e^{-iLp\theta}) \frac{i}{\sqrt{2L}} \sum_{n \in \mathbb{Z} + \frac{1}{2}} \frac{1}{\sqrt{\cosh(\alpha_n)}} \left[\frac{e^{\frac{\theta + \alpha_n}{2}} - e^{-\frac{\theta + \alpha_n}{2}}}{E_n - E_\theta} a_n + \frac{e^{\frac{\theta + \alpha_n}{2}} + e^{-\frac{\theta + \alpha_n}{2}}}{E_n + E_\theta} a_n^\dagger \right] \end{aligned}$$

which gives the result of the limit $\theta \rightarrow \alpha_n$ to be

$$W_-(\theta) \sim \sqrt{2L \cosh(\alpha_n)} a_n. \quad (\text{A.4})$$

Hence we recover (1.8).

B Proof of the equivalence between the finite-temperature form factor expansion and the form factor expansion on the circle

By re-arranging the rapidity variables (along with their associated charges) in (1.6), it is possible to bring the finite-temperature form factor expansion of two-point functions in the form

$$\begin{aligned} \langle \langle \mathcal{O}_1(x, \tau) \mathcal{O}_2(0, 0) \rangle \rangle_L = & \sum_{K=0}^\infty \sum_{k=0}^K \int \frac{d\theta_1 \dots d\theta_K e^{\sum_{j=1}^k (imx \sinh \theta_j - m\tau \cosh \theta_j) - \sum_{j=k+1}^K (imx \sinh \theta_j - m\tau \cosh \theta_j)}}{k!(K-k)! \prod_{j=1}^k (1 + e^{-mL \cosh \theta_j}) \prod_{j=k+1}^K (1 + e^{mL \cosh \theta_j})} \times \\ & \times f_{+, \dots, +, -, \dots, -}^{\mathcal{O}_1}(\theta_1, \dots, \theta_k, \theta_{k+1}, \dots, \theta_K; L) f_{+, \dots, +, -, \dots, -}^{\mathcal{O}_2}(\theta_K, \dots, \theta_{k+1}, \theta_k, \dots, \theta_1; L). \end{aligned} \quad (\text{B.1})$$

If the first field \mathcal{O}_1 is a twist field, we will consider it to have a branch cut on its right (positive x direction); on the other hand, if the second field \mathcal{O}_2 is a twist field, we will consider it to have a branch cut on its left (negative x direction). This ensures that when we obtain the form factor expansion on the circle, the intermediate states are in the sector with anti-periodic

conditions on the fermion fields (whereas the vacuum vectors may be in the sector with periodic conditions). Moreover, we will assume $x > 0$; this insures that the operators are time-ordered in the quantization on the circle.

Consider shifting the contours of integration associated to the rapidity variables $\theta_1, \dots, \theta_k$ towards the positive imaginary direction, by an amount $i\pi$. The only poles that contribute are those from the factors $\prod_{j=1}^k (1 + e^{-mL \cosh \theta_j})$ in the denominator.

Consider the terms arising from taking N poles, with fixed K . They are given by

$$\begin{aligned}
& \sum_{k=N}^K \int \frac{d\theta_{N+1} \cdots d\theta_K e^{\sum_{j=1}^N (-mx \cosh \alpha_{n_j} - im\tau \sinh \alpha_{n_j}) - \sum_{j=N+1}^K (imx \sinh \theta_j - m\tau \cosh \theta_j)}}{k!(K-k)! \prod_{j=N+1}^K (1 + e^{mL \cosh \theta_j})} \times \\
& \times (-1)^{k-N} \frac{k!}{N!(k-N)!} \prod_{j=1}^N \left(\frac{2\pi}{mL \cosh(\alpha_{n_j})} \right) \times \\
& \times f_{+, \dots, +, -, \dots, -}^{\mathcal{O}_1} \left(\alpha_{n_1} + \frac{i\pi}{2}, \dots, \alpha_{n_N} + \frac{i\pi}{2}, \theta_{N+1}, \dots, \theta_K; L \right) \times \\
& \times f_{+, \dots, +, -, \dots, -}^{\mathcal{O}_2} \left(\theta_K, \dots, \theta_{N+1}, \alpha_{n_N} + \frac{i\pi}{2}, \dots, \alpha_{n_1} + \frac{i\pi}{2}; L \right)
\end{aligned} \tag{B.2}$$

where each element of the set $\{n_j, j = 1, \dots, N\}$ is a number in $\mathbb{Z} + \frac{1}{2}$. On the third line, there are N positive charges and $K - N$ negative charges; on the fourth line, there are $K - N$ positive charges and N negative charges. On the second line, the factor $(-1)^{k-N}$ comes from shifting $k - N$ rapidity variables by $i\pi$, and taking the imaginary factors from the crossing relations, (5.1) and Point 4 in sub-section 6.2. The sum over k can be done, and vanishes whenever $N \neq K$. Hence we are left with

$$\begin{aligned}
& \frac{e^{\sum_{j=1}^K (-mx \cosh \alpha_{n_j} - n_j \frac{2\pi i\tau}{L})}}{K!} \prod_{j=1}^K \left(\frac{2\pi}{mL \cosh(\alpha_{n_j})} \right) \times \\
& \times f_{+, \dots, +}^{\mathcal{O}_1} \left(\alpha_{n_1} + \frac{i\pi}{2}, \dots, \alpha_{n_K} + \frac{i\pi}{2}; L \right) f_{-, \dots, -}^{\mathcal{O}_2} \left(\alpha_{n_K} + \frac{i\pi}{2}, \dots, \alpha_{n_1} + \frac{i\pi}{2}; L \right).
\end{aligned} \tag{B.3}$$

When summed over K and over $\{n_j\}$, this reproduces the form factor expansion on the circle as in (1.4) if we use (1.8), up to a phase factor in accordance with (1.3).

C Operators on \mathcal{H} corresponding to twist fields

Let us now turn to the description of the operators associated to twist fields: the order field σ and disorder field μ .

The order and disorder fields can be essentially defined by their OPE's with the fermion fields. In order to keep the proper OPE's, it is convenient to remember that with the conformal normalization $\psi(z)\psi(z') \sim (z-z')^{-1}$, the leading term of the OPE's are $\psi(z)\sigma(0) \sim \sqrt{\frac{i}{2z}}\mu(0)$, $\psi(z)\mu(0) \sim$

$\sqrt{\frac{-i}{2z}}\sigma(0)$. Our normalization of the fermions, however, is given by (2.9). Hence we have

$$\psi(x, \tau)\sigma(0) \sim \frac{i}{2\sqrt{\pi}(x+i\tau)}\mu(0), \quad \psi(x, \tau)\mu(0) \sim \frac{1}{2\sqrt{\pi}(x+i\tau)}\sigma(0) \quad (\text{C.1})$$

and

$$\bar{\psi}(x, \tau)\sigma(0) \sim -\frac{i}{2\sqrt{\pi}(x-i\tau)}\mu(0), \quad \bar{\psi}(x, \tau)\mu(0) \sim \frac{1}{2\sqrt{\pi}(x-i\tau)}\sigma(0). \quad (\text{C.2})$$

By convention, the disorder field has nonzero odd-particle form factors only, and the order field has nonzero even-particle form factors only. In particular, σ has a nonzero real vacuum expectation value.

The finite-temperature correlation functions of fermion fields with insertion of these spin fields are functions on coverings of the cylinder. In order to define operators on the Hilbert space \mathcal{H} corresponding to the twist fields σ and μ , we must choose Riemann sheets. In fact, since the quantization is on the line, the cylindrical geometry is implemented by tracing over \mathcal{H} , so that we may define the operators corresponding to twist fields by specifying the Riemann sheets on the plane (that is, it is sufficient to define them at zero temperature). It will be convenient to define two operators on \mathcal{H} for each twist field, corresponding to different choices of branch cuts. Inside vacuum expectation values in \mathcal{H} , the choice of branch cut does not matter, but inside traces, it does.

Consider the product of fields $(\psi(x, \tau)\mu(0))_+$ defining (inside zero-temperature correlation functions) a function on the plane satisfying the free massive equation of motion everywhere except at the branch cut $\tau = 0, x > 0$ (and except at the position of other local fields, if any). We will define the operator μ_+ by the fact that under the mapping of matrix elements of operators to correlation functions,

$$\begin{aligned} \psi(x, \tau)\mu_+(0) &\mapsto (\psi(x, \tau)\mu(0))_+ \quad (\tau > 0) \\ \mu_+(0)\psi(x, \tau) &\mapsto -(\psi(x, \tau)\mu(0))_+ \quad (\tau < 0) \end{aligned} \quad (\text{C.3})$$

and

$$\begin{aligned} \psi(x, \tau)\mu_+(0) &\mapsto \left[(\psi(x, \tau')\mu(0))_+ \right]_{\tau':0^+ \rightarrow \tau} \quad (\tau < 0) \\ \mu_+(0)\psi(x, \tau) &\mapsto -\left[(\psi(x, \tau')\mu(0))_+ \right]_{\tau':0^- \rightarrow \tau} \quad (\tau > 0). \end{aligned} \quad (\text{C.4})$$

That is, in the second set of maps, the product of operators gives, inside vacuum expectation values, correlation functions that are continued through the branch cut of $(\psi(x, \tau')\mu(0))_+$ if $x > 0$.

Similarly, consider the product of fields $(\psi(x, \tau)\mu(0))_-$ defining a function on the plane that satisfies the free massive equation of motion everywhere except at the branch cut $\tau = 0, x < 0$ (and except at the position of other local fields, if any); this function coincides with the function defined by $(\psi(x, \tau)\mu(0))_+$ when $\tau > 0$. We will define the operator μ_- by the fact that under the mapping of matrix elements of operators to correlation functions,

$$\begin{aligned} \psi(x, \tau)\mu_-(0) &\mapsto (\psi(x, \tau)\mu(0))_- \quad (\tau > 0) \\ \mu_-(0)\psi(x, \tau) &\mapsto -(\psi(x, \tau)\mu(0))_- \quad (\tau < 0) \end{aligned} \quad (\text{C.5})$$

and

$$\begin{aligned}\psi(x, \tau)\mu_-(0) &\mapsto \left[(\psi(x, \tau')\mu(0))_-\right]_{\tau'=0^+ \rightarrow \tau} \quad (\tau < 0) \\ \mu_-(0)\psi(x, \tau) &\mapsto -\left[(\psi(x, \tau')\mu(0))_-\right]_{\tau'=0^- \rightarrow \tau} \quad (\tau > 0) .\end{aligned}\tag{C.6}$$

We make similar definitions for the order field, without the minus sign that represents the odd statistics of μ_\pm with fermion operators. We have:

$$\begin{aligned}\psi(x, \tau)\sigma_+(0) &\mapsto (\psi(x, \tau)\sigma(0))_+ \quad (\tau > 0) \\ \sigma_+(0)\psi(x, \tau) &\mapsto (\psi(x, \tau)\sigma(0))_+ \quad (\tau < 0)\end{aligned}\tag{C.7}$$

and

$$\begin{aligned}\psi(x, \tau)\sigma_+(0) &\mapsto \left[(\psi(x, \tau')\sigma(0))_+\right]_{\tau'=0^+ \rightarrow \tau} \quad (\tau < 0) \\ \sigma_+(0)\psi(x, \tau) &\mapsto \left[(\psi(x, \tau')\sigma(0))_+\right]_{\tau'=0^- \rightarrow \tau} \quad (\tau > 0)\end{aligned}\tag{C.8}$$

as well as

$$\begin{aligned}\psi(x, \tau)\sigma_-(0) &\mapsto (\psi(x, \tau)\sigma(0))_- \quad (\tau > 0) \\ \sigma_-(0)\psi(x, \tau) &\mapsto (\psi(x, \tau)\sigma(0))_- \quad (\tau < 0)\end{aligned}\tag{C.9}$$

and

$$\begin{aligned}\psi(x, \tau)\sigma_-(0) &\mapsto \left[(\psi(x, \tau')\sigma(0))_-\right]_{\tau'=0^+ \rightarrow \tau} \quad (\tau < 0) \\ \sigma_-(0)\psi(x, \tau) &\mapsto \left[(\psi(x, \tau')\sigma(0))_-\right]_{\tau'=0^- \rightarrow \tau} \quad (\tau > 0) .\end{aligned}\tag{C.10}$$

With these definitions and with the OPE's (C.1) and (C.2), it is possible to check that the fields σ_\pm are Hermitian, that μ_+ is Hermitian and that μ_- is anti-Hermitian on \mathcal{H} :

$$\sigma_\pm^\dagger = \sigma_\pm , \quad \mu_\pm^\dagger = \pm \mu_\pm .\tag{C.11}$$

Note that relations (C.3 – C.10) also hold for operators corresponding to descendants of the order and disorder fields.

D Normalization of the one-particle finite-temperature form factors

The normalization can be obtained by considering the OPE's $\psi(x, \tau)\mu_\pm(0)$. Consider first $\mu_-(0)$. We have

$$\langle\langle \mathcal{T}\psi(x, \tau)\mu_-(0) \rangle\rangle_L \sim \frac{1}{2\sqrt{\pi(x+i\tau)}} \langle\langle \sigma \rangle\rangle_L\tag{D.1}$$

where \mathcal{T} means time ordering (the latest operator being placed on the left) and where the square root is taken on its principal branch. We will take $\tau > 0$, $x > 0$ in (D.1); then we can repeat the calculation of Appendix B. Since only the one-particle finite temperature form factors contribute, the results of Appendix B give, for the left-hand side,

$$\sum_{n \in \mathbb{Z} + \frac{1}{2}} e^{-mx \cosh \alpha_n - \frac{2i\pi n\tau}{L}} \frac{2\pi}{mL \cosh(\alpha_n)} f_+^\psi \left(\alpha_n + \frac{i\pi}{2}; L \right) f_-^{\mu-} \left(\alpha_n + \frac{i\pi}{2}; L \right) . \quad (\text{D.2})$$

The leading $x + i\tau \rightarrow 0^+$ behavior is obtained by looking at the leading $|n| \rightarrow \infty$ behavior of the summand. We have

$$f_-^{\mu-} \left(\alpha_n + \frac{i\pi}{2}; L \right) \sim e^{-\frac{i\pi}{4}} C(L) .$$

From (5.3), the main contribution comes from the region $n \rightarrow \infty$ only, and we obtain for the leading behavior of the left-hand side of (D.1),

$$\frac{C(L)}{\sqrt{L}} \sum_{n \in \mathbb{Z} + \frac{1}{2}, n > 0} e^{-\frac{2\pi n(x+i\tau)}{L}} \frac{1}{\sqrt{n}} \sim \sqrt{\frac{1}{2(x+i\tau)}} C(L) . \quad (\text{D.3})$$

Hence, we find (7.7).

It is instructive to repeat the calculation for $\mu_+(0)$ and to verify that $C(L)$ is indeed as given above. We have

$$\langle \langle \mathcal{T} \psi(x, \tau) \mu_+(0) \rangle \rangle_L \sim \frac{1}{2\sqrt{\pi(x+i\tau)}} \langle \langle \sigma \rangle \rangle_L \quad (\text{D.4})$$

where the square root is on a branch that coincides with the principal branch for $\tau > 0$ but that has a cut at $\tau = 0$, $x > 0$. In order to apply the results of Appendix B, we need take $\tau < 0$, $x < 0$, and we have

$$\langle \langle \mu_+(0) \psi(x, \tau) \rangle \rangle_L \sim i \frac{1}{2\sqrt{-\pi(x+i\tau)}} \langle \langle \sigma \rangle \rangle_L \quad (\text{D.5})$$

where now the square root is on its principal branch. This has expansion

$$\sum_{n \in \mathbb{Z} + \frac{1}{2}} e^{mx \cosh \alpha_n + \frac{2i\pi n\tau}{L}} \frac{2\pi}{mL \cosh(\alpha_n)} f_+^{\mu+} \left(\alpha_n + \frac{i\pi}{2}; L \right) f_-^\psi \left(\alpha_n + \frac{i\pi}{2}; L \right) . \quad (\text{D.6})$$

With $f_+^{\mu+}(\theta; L) \sim e^{\frac{i\pi}{4}} C(L)$ and using (5.3), we have, for the leading $x + i\tau \rightarrow 0^-$ behavior of the left-hand side,

$$\frac{i C(L)}{L} \sum_{n \in \mathbb{Z} + \frac{1}{2}, n > 0} e^{\frac{2\pi n(x+i\tau)}{L}} \frac{1}{\sqrt{n}} \sim i \sqrt{\frac{1}{-2(x+i\tau)}} C(L) \quad (\text{D.7})$$

which agrees with (D.5) if $C(L)$ is given by (7.7).

E Fredholm determinant representations for two-point functions at finite temperature

The results of Appendix B essentially show that

$$\begin{aligned}\langle\langle\sigma_+(x, \tau)\sigma_-(0, 0)\rangle\rangle_L &= {}_L\langle\text{vac}|\sigma_L(-\tau, x)\sigma_L(0, 0)|\text{vac}\rangle_L \\ \langle\langle\mu_+(x, \tau)\mu_-(0, 0)\rangle\rangle_L &= {}_L\langle\text{vac}|\mu_L(-\tau, x)\mu_L(0, 0)|\text{vac}\rangle_L\end{aligned}\quad (\text{E.1})$$

where on the right-hand side, the vacuum expectation values are on the circle in the Ramond sector. In a similar fashion, an analysis of the finite-temperature form factor expansion shows that

$$\begin{aligned}\langle\langle\sigma_-(x, \tau)\sigma_+(0, 0)\rangle\rangle_L &= \langle\langle\sigma_+(-x, -\tau)\sigma_-(0, 0)\rangle\rangle_L \\ \langle\langle\mu_-(x, \tau)\mu_+(0, 0)\rangle\rangle_L &= -\langle\langle\mu_+(-x, -\tau)\mu_-(0, 0)\rangle\rangle_L.\end{aligned}\quad (\text{E.2})$$

Note that the second equation is in agreement with the fermionic statistic of the operators μ_\pm .

Using the finite-temperature form factors (7.9) and (7.10) for twist fields, we have then the following large distance expansions of two-point functions in the R sector:

$$\begin{aligned}\langle\langle\sigma_+(x, \tau)\sigma_-(0, 0)\rangle\rangle_L &= \\ \sum_{\substack{k=0 \\ k \text{ even}}}^{\infty} \sum_{\epsilon_1, \dots, \epsilon_k = \pm} \int \frac{d\theta_1 \dots d\theta_k}{k!} \frac{e^{\sum_{j=1}^k \epsilon_j (imx \sinh \theta_j - m\tau \cosh \theta_j)}}{\prod_{j=1}^k (1 + e^{-\epsilon_j mL \cosh \theta_j})} i^k \prod_{j=1}^k (f_{\epsilon_j}^{\mu_+}(\theta_j; L))^2 \prod_{1 \leq i < j \leq k} \tanh\left(\frac{\theta_j - \theta_i}{2}\right)^{2\epsilon_i \epsilon_j}\end{aligned}\quad (\text{E.3})$$

and

$$\begin{aligned}\langle\langle\mu_+(x, \tau)\mu_-(0, 0)\rangle\rangle_L &= \\ - \sum_{\substack{k=0 \\ k \text{ odd}}}^{\infty} \sum_{\epsilon_1, \dots, \epsilon_k = \pm} \int \frac{d\theta_1 \dots d\theta_k}{k!} \frac{e^{\sum_{j=1}^k \epsilon_j (imx \sinh \theta_j - m\tau \cosh \theta_j)}}{\prod_{j=1}^k (1 + e^{-\epsilon_j mL \cosh \theta_j})} i^k \prod_{j=1}^k (f_{\epsilon_j}^{\mu_+}(\theta_j; L))^2 \prod_{1 \leq i < j \leq k} \tanh\left(\frac{\theta_j - \theta_i}{2}\right)^{2\epsilon_i \epsilon_j}.\end{aligned}\quad (\text{E.4})$$

Following [32, 31], Fredholm determinant representations can now easily be obtained from the formulas

$$\det_{i,j} \left\{ \frac{u_i - u_j}{u_i + u_j} \right\} = \begin{cases} \prod_{1 \leq i < j \leq k} \left(\frac{u_i - u_j}{u_i + u_j} \right)^2 & k \text{ even} \\ 0 & k \text{ odd} \end{cases}\quad (\text{E.5})$$

and

$$\det_{i,j} \left\{ \frac{1}{u_i + u_j} \right\} = \frac{1}{2^k u_1 \dots u_k} \prod_{1 \leq i < j \leq k} \left(\frac{u_i - u_j}{u_i + u_j} \right)^2.\quad (\text{E.6})$$

Formula (E.5) gives

$$\langle\langle\sigma_+(x, \tau)\sigma_-(0, 0)\rangle\rangle_L = \det(\mathbf{1} + \mathbf{K})\quad (\text{E.7})$$

where \mathbf{K} is an integral operator with an additional index structure, defined by its action $(\mathbf{K}f)_\epsilon(\theta) = \sum_{\epsilon'=\pm} \int_{-\infty}^{\infty} d\theta' K_{\epsilon, \epsilon'}(\theta, \theta') f_{\epsilon'}(\theta')$ and its kernel

$$K_{\epsilon, \epsilon'}(\theta, \theta') = i(f_{\epsilon}^{\mu_+}(\theta; L))^2 \tanh\left(\frac{\theta' - \theta}{2}\right)^{\epsilon \epsilon'} \frac{e^{\epsilon(imx \sinh \theta - m\tau \cosh \theta)}}{1 + e^{-\epsilon mL \cosh \theta}}.\quad (\text{E.8})$$

Finally, in order to obtain two-point functions of disorder fields, we must consider the linear combinations $\sigma \pm \mu$. Formula (E.6) gives

$$\langle\langle(\sigma_+(x, \tau) + \eta\mu_+(x, \tau))(\sigma_-(0, 0) + \eta\mu_-(0, 0))\rangle\rangle_L = \det(\mathbf{1} + \mathbf{J}^{(\eta)}) \quad (\text{E.9})$$

with $\eta = \pm$ and by definition $(\mathbf{J}^{(\eta)}f)_\epsilon(u) = \sum_{\epsilon'=\pm} \int_0^\infty du' J_{\epsilon, \epsilon'}^{(\eta)}(u, u') f_{\epsilon'}(u')$ where the kernel is given by

$$J_{\epsilon, \epsilon'}^{(\eta)}(u, u') = -2\eta i(f_\epsilon^{\mu+}(\ln(u); L))^2 \frac{1}{\epsilon u + \epsilon' u'} \frac{e^{\frac{\epsilon}{2}(imx(u-u^{-1})-m\tau(u+u^{-1}))}}{1 + e^{-\frac{\epsilon m L}{2}(u+u^{-1})}}. \quad (\text{E.10})$$

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